Incremental method of Young modulus updating procedure in topology optimization

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Abstract. This paper presents a new Young modulus updating procedure as an extension to the SIMP method used for topology optimization. In essence, the modified Young modulus updating procedure consists in taking into account in a given optimization step not only the material density from the preceding step, but also the increment in density in the two preceding steps. Thanks to this, it is possible to obtain a solution in cases in which the classic SIMP method failed. The variational approach was adopted and the structure's strain energy was minimized under constraints imposed on body mass. FEM was used to solve numerical examples. The numerical analysis confirmed the effectiveness of the proposed method, particularly for structures with relatively long spans.

Key words: topology optimization, minimum compliance, new incremental method of updating Young modulus.

1. Introduction

Generally, the aim of topology optimization is to find an optimum distribution of the material from which a given structure is to be made, in a certain defined area, called a design area, under the prescribed boundary conditions and load. The aim of optimization here is to maximize stiffness, which corresponds to minimizing compliance (Π^E). Since there is a mutual correspondence between a structure's compliance and the strain energy accumulated in it ($\Pi^E = 2\Pi^I$), the strain energy minimization problem:

$$\Pi^I = \frac{1}{2} \int_V \varepsilon^T \mathbf{C}\, \varepsilon\, dV \tag{1}$$

is considered under the constraints imposed on body mass, i.e. the quantity of available mass does not change in the course of optimization and in each successive step the same amount of mass is distributed within the design area:

$$m_j = m_0, \tag{2}$$

where

$$m_0 = \alpha\, m, \quad 0 < \alpha < 1 \quad \text{and} \quad m = V \rho_0, \tag{3}$$

j stands for the number of a consecutive optimization step and m_j is the body mass in the optimization step, m_0 – the quantity of available mass, α – a mass reduction coefficient defining what proportion of the mass located in the design area is used in the optimization process, ρ_0 – the density of the material from which the structure is to be built and V – the volume of the design area.

In this paper the well-known SIMP method [1–3] has been modified by changing the way of updating the Young modulus, as described below.

In the classic SIMP method, the Young modulus in a considered point (finite element) of the body depends on the material density in this point in the preceding optimization step (bearing the number j-1):

$$E_j = E_0 \left(\frac{\rho_{j-1}}{\rho_0} \right)^p, \tag{4}$$

where E_j is the Young modulus in the current step (j), E_0 is the Young modulus of the material from which the structure is to be built and ρ_{j-1} is the preceding step material density relative to that of current step j. In the proposed method, the Young modulus in a considered finite element in the j-th optimization step depends on the element's material density in the preceding step and also on the increment in this element's density between steps j-1 and j-2. The increment is additionally scaled through parameter a. The so formulated increment is defined as δ

$$\delta = \frac{\rho_{j-1} - \rho_{j-2}}{a} \tag{5}$$

and it is added or subtracted from the preceding step material density:

$$E_j = E_0 \left(\frac{\rho_{j-1} + \delta}{\rho_0} \right)^p, \tag{6}$$

$$E_j = E_0 \left(\frac{\rho_{j-1} - \delta}{\rho_0} \right)^p. \tag{7}$$

The algorithm with updating according to Eq. (6) is further referred to as Alg+ while the one with updating according to Eq. (7) is referred as Alg−. Since the proposed algorithm uses densities from the step whose number is lower by 2 than the number of the considered step, it can be introduced starting with the step whose number is ($j \geq 2$) according this scheme:

Step 0

$$E_{j=0} = E_0,$$

Step 1

$$E_{j=1} = E_0 \left(\frac{\rho_{j=0}}{\rho_0} \right)^p,$$

Step 2   (8)

$$E_{j=2} = E_0 \left(\frac{\rho_{j=1} \pm \delta}{\rho_0} \right)^p,$$

Step 3 (...)

$$E_{j=3} = E_0 \left(\frac{\rho_{j=2} \pm \delta}{\rho_0} \right)^p, (...).$$

As it is apparent, the proper updating relation is the same for steps no. 2, 3, etc., which is denoted by (...).

This paper concentrates on an analysis of selected numerical results. Since both the topology and the strain energy value characterizing it are critical for the analysis, figures showing topologies with strain energy values should be presented to show how the topologies change depending on the control parameters. However, this would require much space and so only selected results are presented here. The topologies have the form of: a material/void distribution (a) or a distribution where besides the material/void there is also material whose density is slightly varied and slightly lower than that of the material from which the structure is made (b). Graphically, case (a) is represented as black and white distribution and case (b) as a black and white distribution with shades of grey.

2. Numerical examples

A program based on the above algorithm was written in *Matlab*. Then its effectiveness was checked: by adjusting (increasing) parameter a, solutions identical with the classic SIMP solution would be obtained and by reducing this parameter, solutions in the cases in which the classic SIMP method yielded no solution were obtained.

The solutions obtained using the proposed incremental method are compared below with the ones yielded by the classic SIMP method. The latter solutions were obtained using a program called Alg_1 (differing in only the updating procedure from Alg− and Alg+) developed by the authors. Consequently, the comparative analysis is even more reliable and its conclusions can be useful in the further search for more effective topology optimization algorithms.

The Alg_1 algorithm used in the numerical program had been previously tested Ref. [4]. In part it is based on Ref. [5] where among other things threshold functions were used to speed up the process of obtaining optimal solution. Threshold functions define the level of the relatively small strain in the structure's particular areas. On this basis, material is redistributed from the relatively little strained areas to the relatively more strained ones. The idea of introducing threshold functions was derived from the threshold values used in, e.g., [6].

By relating a threshold function to an optimization step number, material can be redistributed gently and smoothly. Thanks to the use of threshold functions one can obtain solutions of the material-void type for low values of exponent p.

The numerical examples provided are typical literature benchmarks, i.e. a freely supported beam and a cantilever. The aim of this approach was to ensure high comparability with the results reported in the literature. A freely supported beam fixed in two ways is considered (Fig. 1). In scheme 1 (*Sch. 1*), two constrains on the left support and one constraint on the right support have been removed. In scheme 2 (*Sch. 2*), two constraints have been removed on each of the supports. The beams are loaded with force P at midspan on the upper edge. The third scheme (*Sch. 3*) represents a cantilever with its left edge fixed, loaded with force P in the middle of its height.
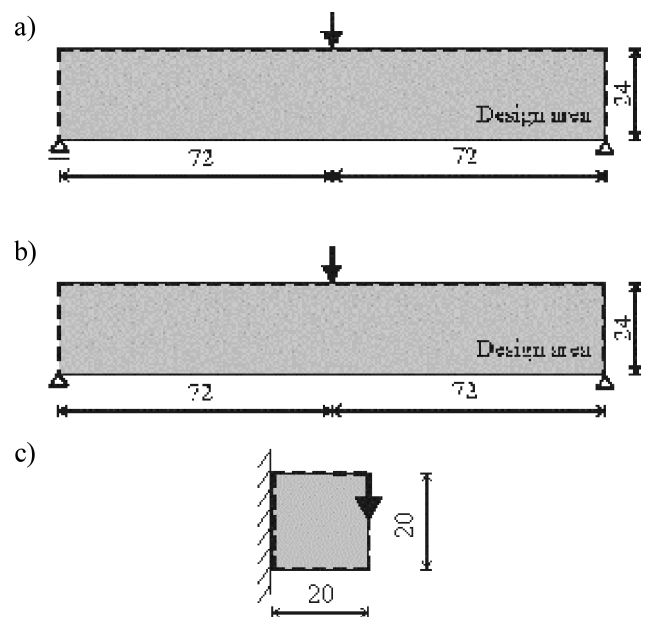

Fig. 1. Static schemes of considered cases: freely supported beam *(Sch. 1 a, Sch. 2 b), and* cantilever c)

The analyzed quantities are expressed as dimensionless. A 144×24 FE mesh was adopted for *Sch. 1* and *Sch. 2* and a 20×20 mesh was used for *Sch. 3*.

Figures 2, 3 and 4 were plotted in such a way that in each of them drawing a) was made on the basis of the results yielded by program Alg_1, while drawing b) contains many results for different parameters, yielded by program Alg+ (Figs. 2 and 3) or Alg− (Fig. 4).

A comparison of strain energy for the cases shown in Fig. 2a and Fig. 2b shows that the Alg+ algorithm yielded strain energy values lower at decreasing parameter a than the ones yielded by Alg_1. The further reduction of parameter a did not lead to a more satisfactory solution. The value of exponent p (Eq. (6) and Eq. (7)), the value of the mass reduction coefficient α are given at the top of the figure. The value of parameter a, the number of the step in which the topology was obtained and the strain energy value (*en*) are shown above each topology.
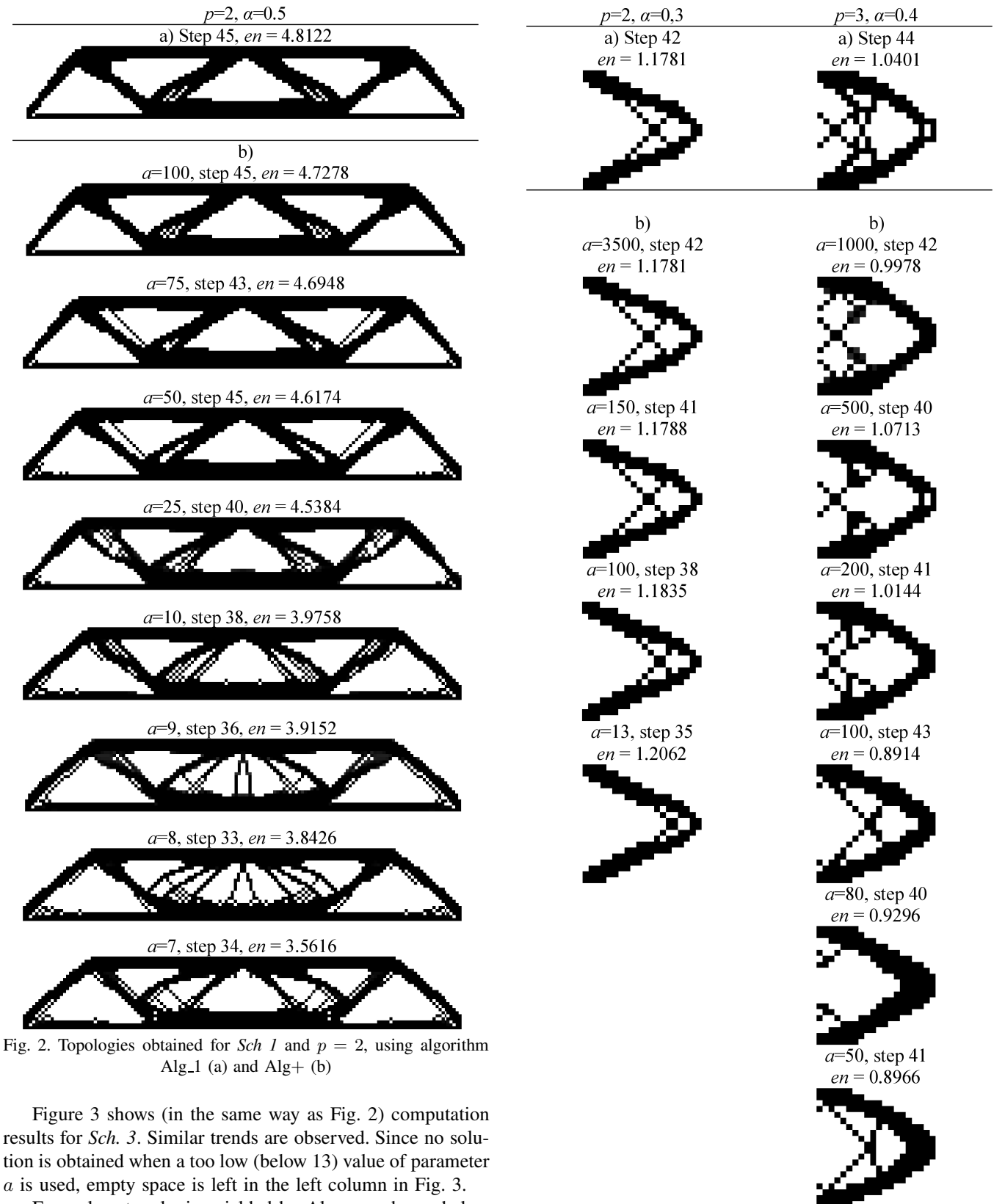
Fig. 2. Topologies obtained for *Sch 1* and $p = 2$, using algorithm Alg_1 (a) and Alg+ (b)

Figure 3 shows (in the same way as Fig. 2) computation results for *Sch. 3*. Similar trends are observed. Since no solution is obtained when a too low (below 13) value of parameter a is used, empty space is left in the left column in Fig. 3.

Exemplary topologies yielded by Alg– are shown below the ones yielded by Alg+. The observed trends concerning topologies and their energies are similar in the two cases.

Fig. 3. Topologies obtained for *Sch. 3* and $p = 2$ and $p = 3$, using algorithm Alg_1 (a) and Alg+ (b)
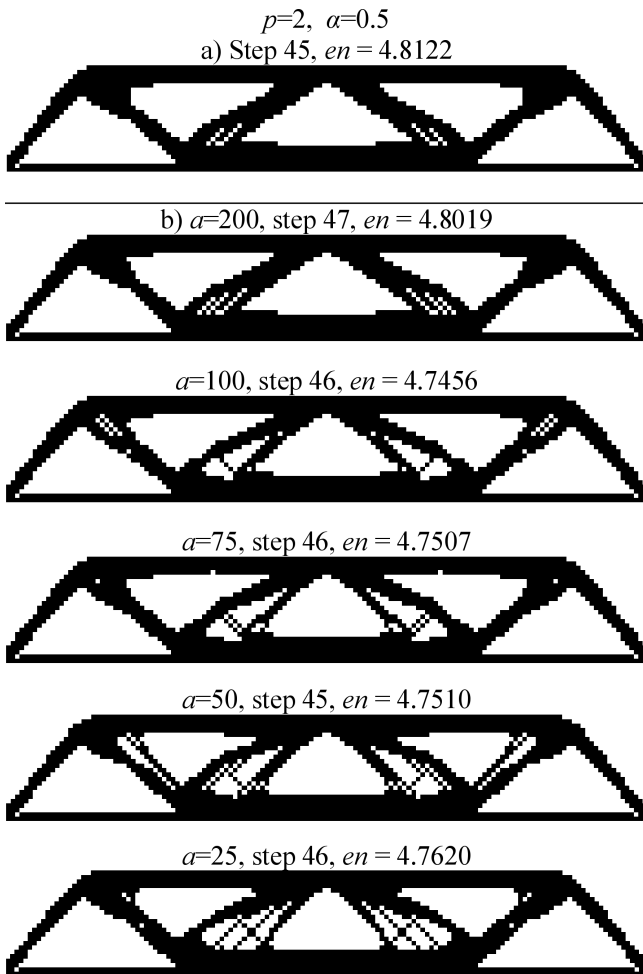
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p=2, *α*=0.5
a) Step 45, *en* = 4.8122

b) *a*=200, step 47, *en* = 4.8019

a=100, step 46, *en* = 4.7456

a=75, step 46, *en* = 4.7507

a=50, step 45, *en* = 4.7510

a=25, step 46, *en* = 4.7620

Fig. 4. Topologies obtained for *Sch. 1* and $p = 2$, using algorithm Alg_1 (a) and Alg− (b)

a)

b)

c)

Fig. 5. Topologies obtained for $p = 4$ and $\alpha = 0.5$, using Alg_1 a), Alg+ b) and Alg− c)

a)

b)
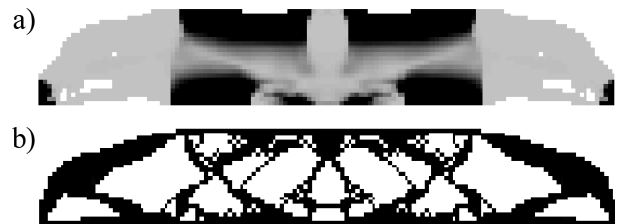
Fig. 6. Topologies obtained for $p = 5$ and $\alpha = 0.45$, using Alg_1 a) and Alg− (step 47) b)

2.1. Results for different exponent values – *p*. In this subsection, exemplary topologies obtained for different values of exponent p in Eqs. (6), (7) are analyzed. First of all, solutions with higher exponent values are subjected to scrutiny since exponents with a value higher than 3 are seldom used in the literature. Higher exponent values result in the relatively quicker removal of material from the relatively less strained areas in which relatively lower strain energy has accumulated. When the incremental method is used, material is removed from the less strained areas not so quickly and precisely as in the case of the classic SIMP method. As an illustration, Fig. 5 shows the solutions obtained for exponent $p = 4$ and mass reduction coefficient $\alpha = 0.5$. Figure 5c shows the topology obtained in step 47 by means of program Alg−. No acceptable topology was obtained for exponent $p = 4$, using either Alg_1 (Fig. 5a) or Alg+ (Fig. 5b). It appears that increment addition or subtraction is of considerable significance since it favourably affects the rate and way of material removal from the less strained elements in the design area. In the cases shown in Fig. 5a and 5b, the topologies were obtained in step 11. No changes were observed in the next steps.

Similar results were obtained for $p = 5$ (Fig. 6). Also here the Alg+ algorithm yielded no solution.
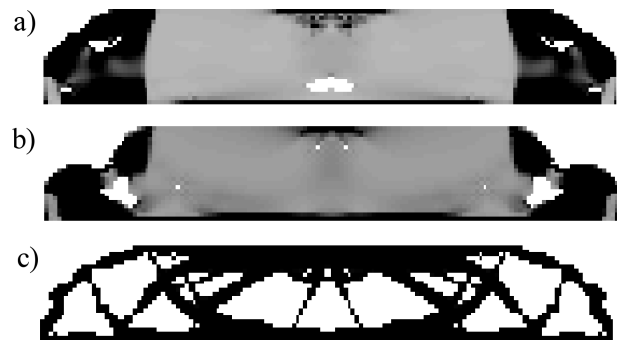
Similar analyses as for the freely supported beam were carried out for cantilever *Sch 3*. The results obtained by means of Alg_1and Alg+, and Alg_1 and Alg− are compared in respectively Fig. 7 and Fig. 8. The more refined topology characterized by lower strain energy, shown in Fig. 7, was obtained using Alg+ and the one in Fig. 8 by means of Alg−.
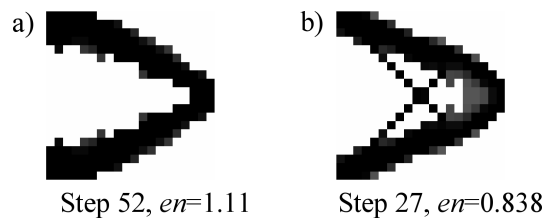
a)   b)

Step 52, *en*=1.11   Step 27, *en*=0.838

Fig. 7. Topologies obtained for $p = 3$ and $\alpha = 0.4$, using Alg_1 a) and Alg+ b)

a)   b)

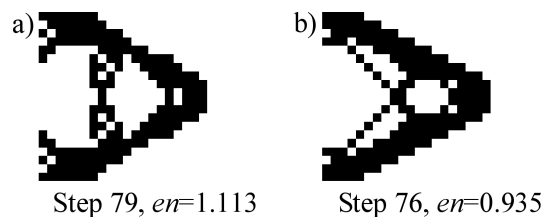

Step 79, *en*=1.113   Step 76, *en*=0.935

Fig. 8. Topologies obtained for $p = 5$ and $\alpha = 0.4$, using Alg_1 a) and Alg− b)

2.2. Results for different FE mesh densities. In order to further check the effectiveness of the incremental algorithms, the latter were tested for different FE mesh densities and the results were compared with the ones obtained using Alg_1. Figure 9 shows the topologies yielded by the Alg_1 algorithm for Sch. 1 under the same control parameters, three different FE mesh densities (72×12, 144×24, 288×48) and a constant 144×24 [m] design area.

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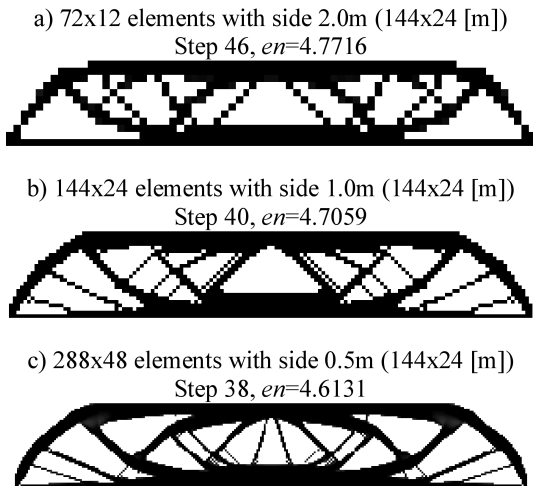
p=1, α=0.5, a=1000

a) 72x12 elements with side 2.0m (144x24 [m])
Step 46, *en*=4.7716

b) 144x24 elements with side 1.0m (144x24 [m])
Step 40, *en*=4.7059

c) 288x48 elements with side 0.5m (144x24 [m])
Step 38, *en*=4.6131

Fig. 9. Topologies obtained using Alg− and *Sch. 1*

p=1, α=0.5, a=1000

a) 72x12 elements with side 2.0m (144x24 [m])
Step 36, *en*=4.7572

b) 144x24 elements with side 1.0m (144x24 [m])
Step 31, *en*=4.5489

c) 288x48 elements with side 0.5m (144x24 [m])
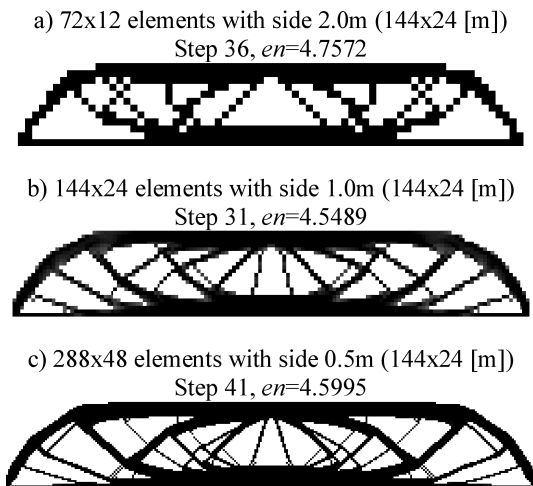Step 41, *en*=4.5995

Fig. 10. Topologies obtained using Alg+ and *Sch. 1*

Similar results obtained by means of Alg+ are shown in Fig. 10. Obviously, the results obtained for the densest mesh have the smoothest shapes. It sometimes happens that the strain energy for the densest mesh solution is somewhat greater than for a less dense mesh. This is due to, among other things, the fact that although the topology obtained in such a case is smooth, it is at the same time more compliant. This

can be sometimes directly seen when one examines the particular structural components (e.g. in the fixing region, where too much material was removed in the course of optimization, whereby the structure is more compliant).

Analyzing the results for all the three schemes one finds that better solution convergence at different mesh densities was achieved when algorithms Alg+ and Alg− were used. Although, as shown above, in one case Alg+ yielded better results while in another case Alg− performed better. This depends on the particular case and the assumed design parameters. In some cases, the initial algorithm yields a slightly better result. Thanks to the increment based on parameter a it is possible to search for better solution than the one yielded by the initial algorithm (Alg_1) while preserving better stability of shape similarity when FE mesh density is changed.

2.3. Results for different ratios of design area sides. Another subject of the analysis were solutions for large spans of beams: Sch. 1, Sch. 2 and Sch. 3. First a 240×24 [m] (a 10:1 side ratio) Sch. 1 beam was analyzed (Fig. 11). The computations were performed for a quite low (for this scheme) value of mass reduction coefficient $\alpha = 0.35$. In the case of the incremental method, controlling parameter $a = 7000$ was used. As it is shown below, the topology yielded by Alg_1 is characterized by the highest strain energy value (due to the unoptimal shape). Algorithm Alg+ yielded a topology with a better arrangement of the branches and a considerably lower strain energy value. The qualitatively best result (with the most well-defined shape and the lowest strain energy value) was yielded by Alg−.

Then a 480×24 (20:1) *Sch. 2* beam was analyzed (Fig. 12). The topologies yielded by respectively Alg_1 and the incremental method for two different control parameter values: $a = 1000$ and $a = 500$ are compared below. In the case of Alg_1, the topology has a poorly refined and unoptimal shape and a very high strain energy value. When Alg+ was used, two better (more openwork) topologies with a lower strain energy value than in the case of Alg_1 were obtained. The topologies yielded by Alg− also show lower strain energy values than the ones yielded by Alg_1. All the topologies are qualitatively good and contain a small number of elements in shades of grey. One should also note how the structure is shaped in the support region, i.e. in the bottom corners of the design area. As the span was increased, a certain amount of material accumulated in the support region.

a) step 46, *en*=38

b) step 27, *en*=26

c) step 40, *en*=25

Fig. 11. Topologies obtained using Alg_1 a), Alg+ b) and Alg− c)

a) Step 43, *en*=68

b) Step 31, *en*=63, *a*=1000

Step 33, *en*=62, *a*=500

c) Step 44, *en*=63, *a*=1000

Step 44, *en*=64, *a*=500
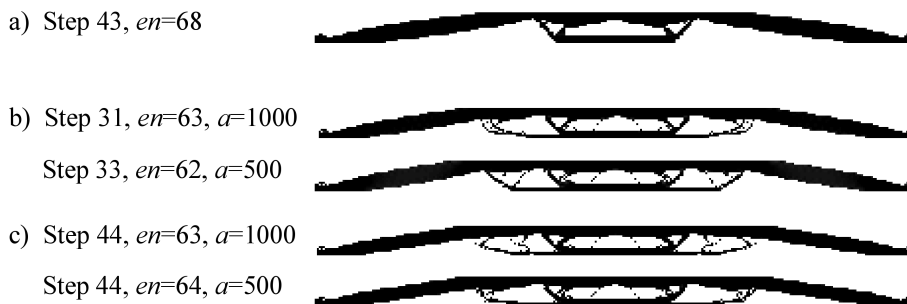
Fig. 12. Topologies obtained using Alg_1 a), and Alg+ b) and Alg− c) ($a = 1000$, $a = 500$)

a) Step 58, *en*=79

b) Step 62, *en*=60

Fig. 13. Topologies for $p = 1$, $a = 1000$ and $\alpha = 0.5$, obtained using Alg_1 a) and Alg− b)

Finally, the cantilever scheme was analyzed. The design area was 120×20 [m], i.e. it had a side ratio of 6:1. Figure 13 shows one of the two considered cases. As one can see, an interesting, original shape of the structure has been obtained and the strain energy value is lower in the case of Alg−-.

To sum up, thanks to the application of the incremental method to design areas in which one of the dimensions is much larger than the other one (e.g. the spans of long bridges of low height or long-reach cantilevers), qualitatively better topologies were obtained. Sometimes the incremental method yielded a topology when the initial algorithm (Alg_1) failed to yield one. It should be noted that the Alg− algorithm is much more computationally effective.

3. Conclusions

The incremental method, which in the Young modulus updating algorithm takes into account a change in mass density in the particular finite elements in the last two steps of the optimization process, represents a significant extension to the SIMP algorithm (Alg_1). The change, i.e. an increment in density, can be taken into account by adding (Alg+) or subtracting (Alg−) the increment value.

The analyses concerning the strain energy level of the obtained topologies, the use of different FE mesh densities and the application of the incremental method to design areas with a large difference in the length of their sides have shown that the use of the incremental method helps to obtain energywise desirable refined shapes. It has also been demonstrated that the use of increments (particularly in the case of Alg−) has a positive effect on computations involving high powers, by improving optimization process convergence. The algorithms (Alg+, Alg−) improved in this way enable the search for qualitatively and energywise better topologies in cases when the initial algorithm fails or yields unoptimal solutions.

To sum up, the proposed incremental method of updating Young's modulus yields better results than the classic SIMP method: the topologies obtained in this way are characterized by a lower strain energy level and are more refined. In particular, better results are obtained for:

1. higher values of the exponent in Eqs. (6) and (7),
2. high ratios of design area sides.

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