Simple analytic conditions for stability of fractional discrete-time linear systems with diagonal state matrix

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Abstract. In the paper the problems of practical stability and asymptotic stability of fractional discrete-time linear systems with a diagonal state matrix are addressed. Standard and positive systems are considered. Simple necessary and sufficient analytic conditions for practical stability and for asymptotic stability are established. The considerations are illustrated by numerical examples.

Key words: linear system, discrete-time, fractional, practical stability, asymptotic stability.

1. Introduction

The problems of analysis and synthesis of dynamic systems described by fractional order differential (or difference) equations have recently attained considerable attention, see [1–8], for example.

In the case of linear continuous-time fractional order systems there are analytic, LMI and frequency domain conditions for asymptotic stability [9–15].

The stability analysis of fractional order discrete-time linear systems is more complicated because asymptotic stability of such systems is equivalent to asymptotic stability of the corresponding infinite-dimensional discrete-time systems of the natural order with delays [16]. Therefore, existing conditions for asymptotic stability are also sufficient.

The conditions for practical stability with a given length of practical implementation for standard fractional discrete-time systems are derived in [16–18]. These conditions are considerable simplified in the case of positive systems [3, 19–21].

The aim of the paper is to give simple analytic necessary and sufficient conditions for practical stability and for asymptotic stability of fractional discrete-time linear systems described by the state-space model with the real diagonal state matrix, standard and positive. To the best knowledge of the author, such conditions have not been established yet.

The following notations are used: $\Re^{n\times m}$ – the set of $n\times m$ real matrices and $\Re^n = \Re^{n\times 1}$; $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with non-negative entries and $\Re_+^n = \Re_+^{n\times 1}$; Z_+ – the set of non-negative integers; I_n – the $n\times n$ identity matrix.

2. Problem formulation

Let us consider the fractional discrete-time linear system described by the homogeneous state equation

$$\Delta^{\overline{\alpha}} x_{i+1} = A x_i, \quad i \in Z_+, \tag{1}$$

where

$$x_i = \begin{bmatrix} x_i^1 \\ \vdots \\ x_i^n \end{bmatrix}, \quad \Delta^{\overline{\alpha}} x_{i+1} = \begin{bmatrix} \Delta^{\alpha_1} x_{i+1}^1 \\ \vdots \\ \Delta^{\alpha_n} x_{i+1}^n \end{bmatrix}, \tag{2}$$

with $x^r \in \Re^{n_r}$, $\alpha_r \in (0,1)$, $r = 1,...,n$, $x_i \in \Re^N$, $N = n_1 + \cdots + n_n$ and

$$A = \begin{bmatrix} A_{11} & ... & A_{1n} \\ \vdots & ... & \vdots \\ A_{n1} & ... & A_{nn} \end{bmatrix}, \quad A_{kr} \in \Re^{n_k \times n_r} \ (k,r = 1,...,n). \tag{3}$$

In (2) $\Delta^{\alpha_r} x_i^r$ denotes the fractional difference of order $\alpha_r \in (0,\ 1)$ of the discrete-time function x_i^r defined by [3]

$$\Delta^{\alpha_r} x_i^r = x_i^r + \sum_{k=1}^{i} (-1)^k \binom{\alpha}{k} x_{i-k}^r, \tag{4a}$$

where

$$\binom{\alpha}{k} = \frac{\alpha_r(\alpha_r - 1)\cdots(\alpha_r - k + 1)}{k!}, \quad k = 1, 2, ... \tag{4b}$$

Using (4) we may write Eq. (1) in the form

$$x_{i+1} = A_0 x_i + \sum_{k=1}^{i} A_k x_{i-k}, \quad i \in Z_+, \tag{5}$$

where

$$A_0 = \begin{bmatrix} A_{11} + I_{n_1}\alpha_1 & ... & A_{1n} \\ \vdots & ... & \vdots \\ A_{n1} & ... & A_{nn} + I_{n_n}\alpha_n \end{bmatrix}, \tag{6a}$$

$$A_k = \begin{bmatrix} I_{n_1} c_k(\alpha_1) & ... & 0 \\ \vdots & ... & \vdots \\ 0 & ... & I_{n_n} c_k(\alpha_n) \end{bmatrix}, \tag{6b}$$

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and

$$c_k(\alpha) = (-1)^k \begin{pmatrix} \alpha \\ k+1 \end{pmatrix}, \qquad k = 1, 2, ... \qquad (7)$$

Equation (5) describes the discrete-time linear system with increasing numbers of delays.

The coefficients (7) can be computed by the following simple algorithm suitable for computer programming [20]

$$c_{k+1}(\alpha_r) = c_k(\alpha_r)\frac{k+1-\alpha_r}{k+2}, \qquad k = 1, 2, ..., \quad (8)$$

where $c_1(\alpha_r) = 0.5\alpha_r(1 - \alpha_r)$.

From (8) it follows that $c_k(\alpha_r) > 0$ for $\alpha_r \in (0, \ 1)$ and $k = 1, 2, ...$ Moreover, coefficients $c_k(\alpha_r)$ strongly decrease for increasing k. Therefore, in practical problems it is assumed that k is bounded by some natural number L. This number is called the length of practical implementation. In this case Eq. (5) takes the form

$$x_{i+1} = \begin{cases} A_0 x_i + \sum_{k=1}^{i} A_k x_{i-k} \text{ for } i = 0, 1, \cdots, L \\ A_0 x_i + \sum_{k=1}^{L} A_k x_{i-k} \text{ for } i = L+1, L+2, \cdots \end{cases} \quad (9)$$

with the initial condition $x_0 \in \Re^N$. Equation (9) describes a linear discrete-time system with L delays in a state.

The time-delay system (9) is called the practical realization of the fractional system (1).

Definition 1 [19]. The fractional system (1) is called practically stable if the system (9) is asymptotically stable.

Definition 2 [19]. The fractional system (1) is called asymptotically stable if the system (9) is asymptotically stable for $L \to \infty$.

From Definition 1 and theory of asymptotic stability of a discrete-time linear system we have the following theorem.

Theorem 1. The fractional system (1) with a given length L of practical implementation is practically stable if and only if

$$w(z) \neq 0, \qquad |z| \geq 1, \qquad (10)$$

where

$$w(z) = \det\left\{I_N z - A_0 - \sum_{k=1}^{L} A_k z^{-k}\right\}, \qquad (11a)$$

or equivalently (for $z \neq 0$),

$$w(z) = \det\left\{I_N z^{L+1} - A_0 z^L - \sum_{k=1}^{L} A_k z^{L-k}\right\}. \quad (11b)$$

From the above it follows that to practical stability checking of the fractional system (1) we can apply the existing methods for the asymptotic stability analysis of the discrete-time systems (9) with delays.

The problem of practical stability of the fractional system (1) (asymptotic stability of time-delay system (9)) in the case $n_1 = \cdots = n_n = 1$ and $N = n$ has been considered in [16–18] for standard systems (i.e. non-positive) and

in [3, 19–21] for positive systems. In [20] it has been shown that practical stability and asymptotic stability of the positive fractional system are equivalent to asymptotic stability of the corresponding natural order positive discrete-time systems without delays of the same size as the system (1). In the case of standard systems there are only sufficient conditions for asymptotic stability [16, 18]. This follows from the fact that asymptotic stability of a fractional discrete-time system is equivalent to asymptotic stability of the corresponding infinite-dimensional discrete-time systems with delays of a natural order [16].

In this paper we consider the fractional system (1) with the state matrix of the form

$$A = \text{block diag}\{\widetilde{A}_1, \ \widetilde{A}_2, \ \cdots, \widetilde{A}_n\}, \qquad (12a)$$

where

$$\widetilde{A}_r = \text{diag}\{a_{r1}, \ a_{r2}, \ \cdots, a_{rn_r}\} \qquad (12b)$$

with not necessarily different entries, i.e. may be $a_{rk} = a_{ij}$ for some $r, i = 1, ..., n$ and $k = 1, ..., n_r$, $j = 1, ..., n_i$.

In this case the block diagonal matrix A_0 has the form

$$A_0 = \begin{bmatrix} \widetilde{A}_1 + I_{n_1}\alpha_1 & ... & 0 \\ \vdots & ... & \vdots \\ 0 & ... & \widetilde{A}_n + I_{n_n}\alpha_n \end{bmatrix}. \quad (13)$$

The matrix A whose all elementary divisors are of first degree ([22], pp. 59–65) can be transformed by the similarity transformation to the diagonal form (12). In this case

$$\text{rank}[\lambda_i I_N - A] = N - n_i,$$

where n_i denotes multiplicity of eigenvalue λ_i, $i = 1, 2, ..., p$ $(p \leq N)$ of the matrix A.

In particular, the matrix A with real distinct eigenvalues can be always transformed to the diagonal form (12).

The aim of the paper is to give simple algebraic necessary and sufficient conditions for practical stability and for asymptotic stability of the fractional system (1), (12), standard and positive.

3. Solution to the problem

From a structure of the matrix A (12) it follows that the system (1) can be written as a set of N scalar fractional systems

$$\Delta^{\alpha_r} x_{i+1}^r = a_{rk} x_i^r, \qquad \alpha_r \in (0, \ 1), \qquad (14)$$

for $r = 1, 2, ..., n$, $k = 1, 2, ..., n_r$.

Therefore, firstly we consider the problems of practical stability and asymptotic stability of a scalar fractional system

$$\Delta^{\alpha} x_{i+1} = ax_i, \qquad \alpha \in (0, \ 1). \qquad (15)$$

The problem of practical stability of the system (15) has been recently considered in [17].

3.1. Stability of scalar system. Practical implementation for $i > L$ of the system (15), of the form,

$$x_{i+1} = (a + \alpha)x_i + \sum_{k=1}^{L} c_k(\alpha)x_{i-k} \qquad (16)$$

has the characteristic function

$$w(z) = z - a - \alpha - \sum_{k=1}^{L} c_k(\alpha)z^{-k}. \qquad (17)$$

We consider the following problem: find values of the coefficient $a \in \Re$ for which the system (15) is: 1) practically stable with fixed length L of practical implementation, 2) asymptotically stable. To solution of this problem we apply the D-decomposition method of Nejmark [23].

According to this method, the real axis $\Re$ is divided by the real root boundary of polynomial (17) on three intervals.

The real roots boundary corresponds to such values of a for which the polynomial (17) has roots $z = 1$ and $z = -1$.

Solving with respect to a the equations $w(1) = 0$ and $w(-1) = 0$, where $w(z)$ has the form (17) one obtains, respectively,

$$a = g(L, \alpha) \quad \text{and} \quad a = b(L, \alpha), \qquad (18)$$

where

$$g(L, \alpha) = 1 - \alpha - \sum_{k=1}^{L} c_k(\alpha), \qquad (19)$$

$$b(L, \alpha) = -1 - \alpha - \sum_{k=1}^{L} c_k(\alpha)(-1)^{-k}. \qquad (20)$$

Hence, we have three intervals for values of the coefficient a:

$$a < b(L, \alpha), \qquad (21a)$$

$$b(L, \alpha) < a < g(L, \alpha), \qquad (21b)$$

$$a > g(L, \alpha). \qquad (21c)$$

Lemma 1. The fractional system (15) with the given length L of practical implementation is practically stable if and only if the condition (21b) holds.

Proof. According to the D-decomposition method, it is sufficient to prove that in the interval $[b(L, \alpha), \; g(L, \alpha)]$ there exists at least one value of a for which the fractional system (15) is practically stable (the system (16) is asymptotically stable). It is easy to check that (see also Fig. 1) $0 \in [b(L, \alpha), \; g(L, \alpha)]$ for all $\alpha \in (0, \; 1)$ and for any finite L. Therefore, for simplicity of considerations, we may choose $a = 0$. In this case, using (17) for $a = 0$, the characteristic equation can be written in the form

$$z^{L+1} - \alpha z^L - \sum_{k=1}^{L} c_k(\alpha)z^{L-k} = 0. \qquad (22)$$

In [24] it has been shown that all roots of the polynomial $z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ have absolute values less

than 1 if $1 > |a_{n-1}| + \cdots + |a_1| + |a_0|$. This condition for Eq. (22) has the form

$$1 > \alpha + \sum_{k=1}^{L} c_k(\alpha). \qquad (23)$$

Using the formula [3, 20]

$$\sum_{k=1}^{\infty} c_k(\alpha) = 1 - \alpha, \qquad \alpha \in (0, \; 1), \qquad (24)$$

from (23) we obtain

$$1 - \alpha - \sum_{k=1}^{L} c_k(\alpha) > 1 - \alpha - \sum_{k=1}^{\infty} c_k(\alpha) = 0.$$

Hence, the condition (23) holds and the characteristic function (17) for $a = 0$ has $L+1$ roots which satisfy the condition $|z_r| < 1$ $(r = 1, 2, ..., L + 1)$. This means, according to the D-decomposition method, that the interval $[b(L, \alpha), \; g(L, \alpha)]$ is the asymptotic stability region of the system (16) and also the practical stability region of the fractional system (15). This completes the proof.

Using (19) and (20) with fixed L, we can find boundaries of practical stability regions in the plane (a, α). The boundaries $g(L, \alpha)$ and $b(L, \alpha)$, computed for $L = 10$, $L = 1000$ and $L = 100\,000$, are shown in Fig. 1.
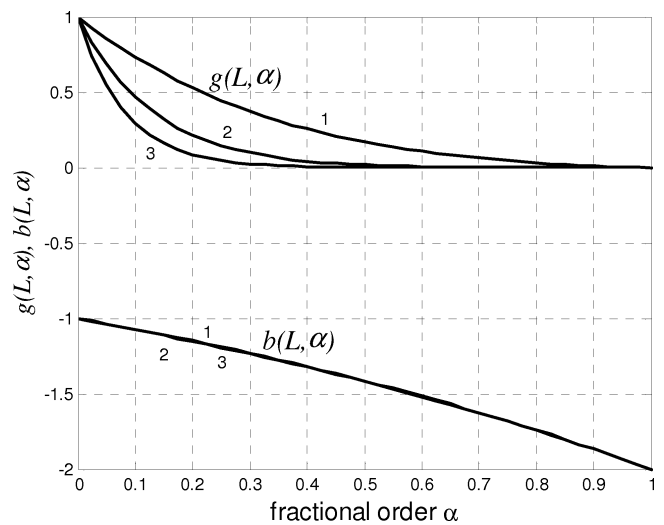

Fig. 1. Boundaries of practical stability regions for $L = 10$ (lines 1), $L = 1000$ (lines 2) and $L = 100\,000$ (lines 3) versus fractional order $\alpha \in (0, \; 1)$

From Fig. 1 it follows that values of $g(L, \alpha)$ depend on the length L of practical implementation whereas values of $b(L, \alpha)$ are almost the same for all $L \geq 10$.

If, for example, $\alpha = 0.1$ then from (21b) and (19), (20) we have the following values of a for which the system (15) is practically stable with the given length L of practical implementation:

- if $L = 10$ then $-1.0750 < a < 0.7333$,
- if $L = 1000$ then $-1.0718 < a < 0.4689$,
- if $L = 100\,000$ then $-1.0718 < a < 0.2959$.

Hence, the system (15) with $a = 0.5$ is practically stable for $L = 10$ but it is not practically stable for any $L \geq 1000$.

From Fig. 1 and (21b), (19), (20) it follows that if $a \in (-1, 0)$ then the system (15) is practically stable independently of the length of practical implementation for any $\alpha \in (0, 1)$ (i.e. it is practically stable for all $L \geq 1$).

It is easy to see that if $\alpha \to 0$ then Eq. (15) takes the form $x_{i+1} = ax_i$ and the system described by this equation is asymptotically stable if and only if $|a| < 1$. Moreover, if $\alpha \to 1$ then from (4a) and (15) we obtain the equation $x_{i+1} = (1 + a)x_i$ and the asymptotic stability condition has the form $-2 < a < 0$. These conditions also follow from Fig. 1.

To establish the conditions for asymptotic stability of the system (15), we consider the condition (21b) and formulae (19), (20) for $L \to \infty$.

Using (24), from (19) one obtains

$$g_\infty(\alpha) = \lim_{L \to \infty} g(L, \alpha) = 0. \qquad (25)$$

Now, we consider the following equality (for $\alpha > 0$ and $|y| \leq 1$)

$$(1 + y)^\alpha = 1 + \alpha y + \frac{\alpha(\alpha - 1)}{2!} y^2 + \cdots$$
$$+ \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)}{k!} y^k + \cdots$$

For $y = 1$ we have

$$2^\alpha = 1 + \alpha + \frac{\alpha(\alpha - 1)}{2!} + \cdots$$
$$+ \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)}{k!} + \cdots \qquad (26)$$

From (7) and (26) it follows that

$$2^\alpha = 1 + \alpha + \sum_{k=1}^{\infty} (-1)^k c_k(\alpha). \qquad (27)$$

Hence, from (20) we have

$$b_\infty(\alpha) = \lim_{L \to \infty} b(L, \alpha) = -2^\alpha. \qquad (28)$$

From the above, formula (21b) and Definition 2 we have the following lemmas.

Lemma 2. The fractional system (15) is asymptotically stable if and only if

$$-2^\alpha < a < 0. \qquad (29)$$

Lemma 3. If
1) $-1 < a < 0$ then the fractional system (15) is asymptotically stable for any $\alpha \in (0, 1)$
2) $-2 < a < -1$ then the fractional system (15) is asymptotically stable for

$$\log_2(-a) < \alpha < 1, \qquad (30)$$

where $\log_2 a$ is the base 2 logarithm of a.

Proof. It follows directly from Lemma 2 and the relationship $-2^\alpha \in (-2, -1)$ for all $\alpha \in (0, 1)$.

If the system (15) is positive, then $x_i \geq 0$ for all $i \in Z_+$ and for any non-negative initial condition x_0. It is well known that the system (15) is positive if and only if [3, 21]

$$a + \alpha \geq 0. \qquad (31)$$

Lemma 4. The positive fractional system (15) with the given length L of practical implementation is practically stable if and only if

$$-\alpha \leq a < g(L, \alpha), \qquad (32)$$

where $g(L, \alpha)$ is defined by (19). Moreover, this system is asymptotically stable if and only if

$$-\alpha \leq a < 0. \qquad (33)$$

Proof. If is easy to see that $b(L, \alpha) < -\alpha$, where $b(L, \alpha)$ is defined by (20). Hence, the condition (32) directly follows from (31) and (21b). Since $-\alpha > -2^\alpha$ for any $\alpha \in (0, 1)$, from (29) and (31) we obtain (33).

From comparison (29) and (33) it follows that the conditions of asymptotic stability of standard and positive fractional order system (15) are different.

3.2. Stability of the system with diagonal matrix. Now we consider the stability problem of the fractional system (1) with diagonal state matrix (12). Stability of this system is equivalent to stability of N scalar systems (14) for $r = 1, 2, ..., n$, $k = 1, 2, ..., n_r$.

Applying Lemmas 1, 2 and 3 to the fractional system (14) one obtains the following theorems.

Theorem 2. The fractional system (1), (12) with the given length L of practical implementation is practically stable if and only if the condition

$$b(L, \alpha_r) < a_{rk} < g(L, \alpha_r) \qquad (34)$$

holds for all $r = 1, 2, ..., n$, $k = 1, 2, ..., n_r$, where

$$g(L, \alpha_r) = 1 - \alpha_r - \sum_{k=1}^{L} c_k(\alpha_r), \qquad (35)$$

$$b(L, \alpha_r) = -1 - \alpha_r - \sum_{k=1}^{L} c_k(\alpha_r)(-1)^{-k}. \qquad (36)$$

Theorem 3. The fractional system (1), (12) is asymptotically stable if and only if the condition

$$-2^{\alpha_r} < a_{rk} < 0 \qquad (37)$$

holds for all $r = 1, 2, ..., n$, $k = 1, 2, ..., n_r$. Moreover, if $a_{rk} \in (-1, 0)$ for all $r = 1, 2, ..., n$ and $k = 1, 2, ..., n_r$, then the system (1), (12) is asymptotically stable for all $\alpha_r \in (0, 1)$.

Theorem 4. If all entries a_{rk} ($r = 1, 2, ..., n$, $k = 1, 2, ..., n_r$) of the matrix (12) are known then the system (1), (12) is asymptotically stable for

$$\overline{\alpha}_r < \alpha_r < 1, \qquad (38)$$

where $\overline{\alpha}_r = 0$ for $a_{rk} \in (-1, 0)$ and $\overline{\alpha}_r = \log_2(-a_{rk})$ for $a_{rk} \in (-2, -1)$.

The system (1), (12) is positive, if $x_i \in \Re_+^N$ for $i \in Z_+$ and for any non-negative initial condition x_0. It is easy to see that positivity of all subsystems (14) for $r = 1, 2, ..., n$, $k = 1, 2, ..., n_r$ is equivalent to positivity of the system (1), (12).

Hence, from Lemma 4 we have the following theorem.

Theorem 5. The positive fractional system (1), (12) with given length L of practical implementation is practically stable if and only if the condition

$$-\alpha_r \leq a_{rk} < g(L, \alpha_r) \qquad (39)$$

holds for all $r = 1, 2, ..., n$, $k = 1, 2, ..., n_r$, where $g(L, \alpha_r)$ is defined by (35). Moreover, this system is asymptotically stable if and only if

$$-\alpha_r \leq a_{rk} < 0 \qquad (40)$$

for all $r = 1, 2, ..., n$, $k = 1, 2, ..., n_r$.

4. Illustrative examples

Example 1. Consider the fractional system (1) with the state matrix (12) with $n = 3$, $n_1 = 2$, $n_2 = n_3 = 1$, $N = 4$, $\alpha_1 = 0.9$, $\alpha_2 = 0.5$, $\alpha_3 = 0.2$ and

$$A_1 = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \quad A_2 = [a_2], \quad A_3 = [a_3]. \qquad (41)$$

Find values of coefficients a_1, a_2 and a_3 for which the system is asymptotically stable.

In this case we have $a_{11} = a_1$, $a_{12} = a_2$, $a_{13} = a_2$ and $a_{14} = a_3$.

Using Theorem 3 one obtains the following inequalities: $-2^{\alpha_1} < a_1 < 0$, $-2^{\alpha_1} < a_2 < 0$, $-2^{\alpha_2} < a_2 < 0$ and $-2^{\alpha_3} < a_3 < 0$. Hence, the system is asymptotically stable if and only if

$$-2^{0.9} < a_1 < 0, \quad -2^{0.5} < a_2 < 0, \quad -2^{0.2} < a_3 < 0.$$

Example 2. Consider the fractional system (1) with the state matrix (12) with $n = 3$, $n_1 = n_2 = n_3 = 1$, $N = 3$ and $A_1 = a_{11} = -0.6$, $A_2 = a_{21} = -1.3$, $A_3 = a_{31} = -1.9$. Find values of fractional orders α_1, α_2 and α_3 for which the system is asymptotically stable.

From Theorem 4 we have: $0 < \alpha_1 < 1$, $\log_2(1.3) = 0.3785 < \alpha_2 < 1$, $\log_2(1.9) = 0.9260 < \alpha_3 < 1$. Hence, the fractional system is asymptotically stable if and only if $\alpha_1 \in (0, 1)$, $\alpha_2 \in (0.3785, 1)$ and $\alpha_3 \in (0.9260, 1)$.

Example 3. Consider the fractional system (1) with $n_1 = n_2 = 1$, $N = 2$, $\alpha_1 = 0.2$, $\alpha_1 = 0.7$ and the state matrix

$$A = \begin{bmatrix} -0.9 & 0 \\ 0 & -0.6 \end{bmatrix}. \qquad (42)$$

In [18] on the basis of the sufficient condition it was shown that the system with the length $L = 25$ of practical implementation is practically stable.

From Theorem 4 directly follows, that the system (1), (42) is asymptotically stable if and only if $\alpha_1 \in (0, 1)$ and $\alpha_2 \in (0, 1)$ and therefore it is practically stable for

all $\alpha_1 \in (0, 1)$, $\alpha_2 \in (0, 1)$ and for all fixed values L of the length of practical implementation.

Example 4. Consider the fractional system (1) with $n_1 = n_2 = n_3 = 1$, $N = 3$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ and the state matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.44 & -3.96 & -3.5 \end{bmatrix}. \qquad (43)$$

Find values of fractional order α for which the system is asymptotically stable.

The matrix (43) has three real eigenvalues $z_1 = -0.8$, $z_2 = -1.2$, $z_3 = -1.5$. By similarity transformation the system (1), (43) can be transformed to the system with diagonal state matrix

$$A_d = \begin{bmatrix} -0.8 & 0 & 0 \\ 0 & -1.2 & 0 \\ 0 & 0 & -1.5 \end{bmatrix}. \qquad (44)$$

From Theorem 4 one obtains: $0 < \alpha_1 = \alpha < 1$, $\log_2(1.2) = 0.1823 < \alpha_2 = \alpha < 1$, $\log_2(1.5) = 0.4055 < \alpha_3 = \alpha < 1$. Hence, the system is asymptotically stable if and only if $\alpha \in (0.4055, 1)$.

5. Concluding remarks

The problems of practical stability and asymptotic stability of discrete-time linear system (1) of fractional order $0 < \alpha < 1$ and diagonal state matrix (12) have been addressed. Simple necessary and sufficient analytic conditions for practical stability and for asymptotic stability of standard systems (Theorem 2 and Theorems 3, 4) and for positive systems (Theorem 5) have been established.

The proposed conditions are also true for the standard fractional discrete-time linear systems which state matrix can be transformed by a similarity transformation to the diagonal form (12). The transformation matrix should be monomial in the case of positive systems.

The main results of the paper may be used in synthesis of the state feedback control $u_i = -Kx_i$ for the system $\Delta^{\overline{\alpha}} x_{i+1} = Ax_i + Bu_i$ such that the closed-loop system is asymptotically stable with diagonal state matrix $A_c = A - BK$.

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