

# Mathematical description of rheological properties of asphalt-aggregate mixes

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**Abstract.** The procedure of the formulation of constitutive equations for asphalt-aggregate mixes is based very often on rheological schemes composed of classical elastic, plastic and viscous elements. The parameters of these schemes can be obtained based on laboratory experiments. In order to obtain better curve fitting results one can use non-classical viscoelastic elements described by fractional derivatives. In this paper we present the characteristics of the fractional viscoelastic Huet-Sayegh model as well as the characteristics of an original simplified fractional model. The results have been obtained using algorithms of numerical calculation of inverse Laplace transforms. Then the proposal of an original rheological model including plasticity has been given. The non-linear differential constitutive relationships of such a model are presented in the paper. The results of computer simulations are also visualized. Finally, 3D viscoelasticplastic models of asphalt aggregate-mixes are proposed. The models are based on a generalized macroscopic theory taking into account the effect of pressure-dependency on yielding.

**Key words:** asphalt-aggregate mixes, rheological models, fractional derivatives, constitutive equations.

## 1. Introduction

Nowadays, road constructions are subjected to extremely high traffic loads. Such factors as traffic densities, axle loads and tire pressures have been increasing in most countries in the last decades. Thus, the optimization of pavement materials is very important in order to avoid damages and subsequently minimize costs for the road construction and maintenance. On the other hand, a pavement failure is not only caused by traffic loads, but is influenced by many other factors like climatic influences and inadequate planning or construction, as well. These factors are able to increase significantly the effects of traffic on the pavement.

In Poland one of the largest infrastructure components are asphalt concrete pavements. Designing such structures needs realistic constitutive models to be taken into consideration. Developing a realistic mathematical model of asphalt-aggregate mixture is a complicated problem. The complexity is attributed to the time-dependency of the binder, the complex nature of temperature effects, plastic flow of the binder, friction among aggregate particles and coupling the above mentioned effects.

The mechanistic design procedures of flexible and semi-rigid pavements are based on the analysis of stresses and strains in critical points of the structure [1]. Having calculated these values one can evaluate the fatigue resistance based on empirical formulas. The fundamental problem in the procedure of pavement design is to elaborate the appropriate constitutive model suited for the structural behaviour modelling within a wide range of mechanical and environmental loadings. The elastic behaviour of asphalt-aggregate mixes com-

poses with its viscous, plastic and fracturing response. Thus, determining the parameters of this material is extremely complicated. Even in such a case when plasticity and fracture are not considered, material viscosity causes temperature and strain rate dependence of the stiffness. For that reason, temperature and strain rates are sometimes integrated in the elastic parameters of the models.

Constitutive equations for asphalt-aggregate mixes are formulated very often using classical rheological models consisting of springs and dashpots [2–4]. The simplest models are the Kelvin-Voigt and Maxwell models consisting of spring and dashpot in parallel and in series, respectively. A more accurate model of viscoelastic behaviour for asphaltic materials is the Burgers model, which consists of the Maxwell model combined in series with the Kelvin-Voigt model. It is possible to express the constitutive equation of viscoelastic materials in terms of fractional order derivatives of stress and strain [5, 6]. Such formulation leads to so called fractional rheological models. Applying this formulation it is possible to obtain better curve fitting results in the procedure of parameters' evaluation based on experimental data.

In this paper we present the characteristics of the fractional Huet-Sayegh (1965) model as well as the characteristics of an original simplified fractional model. We show the results of calculations in the form of creep and relaxation curves as well as hysteretic loops. The characteristics of the fractional models are compared with the characteristics of the classical viscoelastic Burgers model. A special attention is put in the paper on modelling of permanent deformations caused by creep and plasticity. The resulting non-linear models are mathematically described by the systems of explicit differen-

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tial equations. An original viscoelastoplastic model based on generalization of the classical Burgers scheme is presented in details. Finally 3D viscoelastic-plastic models of asphalt aggregate-mixes are proposed. The models are based on a generalized macroscopic theory taking into account the effect of pressure-dependency on yielding.

## 2. Linear rheological elements

Rheological properties of viscoelastic materials can be modelled applying the stress relaxation function  $G$ , which determines the time history of the stress excited by a unit step change in strain [2, 3]. Thus, the relaxation function is a step characteristic of rheological scheme. Applying the relaxation function one can define an integral operator assigning the stress function  $\sigma$  to the differentiable strain function  $\varepsilon$  what leads to the following equation

$$\sigma(t) = G(t) \varepsilon(0) + \int_0^t G(t-\tau) \dot{\varepsilon}(\tau) d\tau. \quad (1)$$

The above equation represents the integral form of the constitutive equation for a viscoelastic solid. The equation is also referred to as a hereditary or convolution integral.

Let us assume the following stress relaxation function defining so called fractional rheological element [6]

$$G(t) := \eta \frac{1}{\Gamma(1-\alpha) t^\alpha}; \quad \alpha \in (0, 1), \quad (2)$$

where  $\eta$  denotes a material parameter and  $\Gamma$  denotes Gamma function

$$\Gamma(1-\alpha) := \int_0^\infty t^{-\alpha} e^{-t} dt. \quad (3)$$

Constitutive relationships of the fractional rheological element may be expressed in the differential form

$$\sigma(t) := \eta D^\alpha \varepsilon(t); \quad \alpha \in (0, 1), \quad (4)$$

where  $D^\alpha \equiv \frac{d^\alpha}{dt^\alpha}$  denotes the  $\alpha$ -th derivative operator.

Applying the Remann-Liouville formula of the fractional order derivative leads to the following definition [6, 7]

$$D^\alpha \varepsilon(t) := \frac{\varepsilon(0)}{\Gamma(1-\alpha) t^\alpha} + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{\varepsilon}(\tau)}{(t-\tau)^\alpha} d\tau. \quad (5)$$

The creep compliance function of the fractional element is defined as follows

$$J(t) := \frac{t^\alpha}{\eta \Gamma(1+\alpha)}; \quad \alpha \in (0, 1). \quad (6)$$

Comparative visualization of elastic, fractional and viscous elements is shown in Fig. 1. Along to the graphical symbols the operator descriptions are given based on Eq. (4).

The constitutive description of the fractional element shows that this element exhibits both elastic and viscous properties depending on the value of  $\alpha \in (0, 1)$ . For  $\alpha \rightarrow 0$  the elastic properties are dominant while in case of  $\alpha \rightarrow 1$  the element behaves like a viscous dashpot.

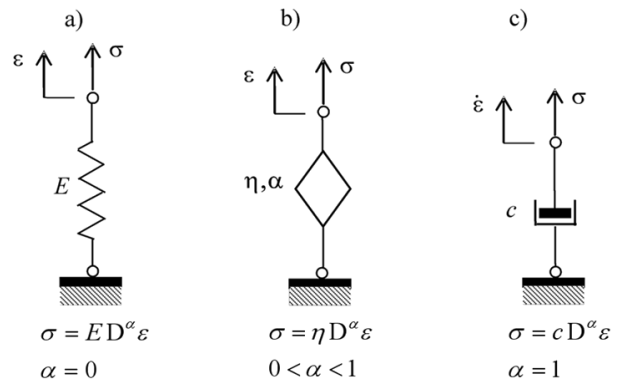


Fig. 1. Comparative visualization of elastic (a), fractional (b) and viscous (c) elements

Figure 2 presents viscoelastic rheological models suited for constitutive modelling of asphalt-aggregate mixes. The classical Burgers model is shown in Fig. 2a while the fractional Huet-Sayegh (1965) model is presented in Fig. 2b. The third fractional model shown in Fig. 2c is our original proposal. It is composed of two branches. The left branch contains spring and fractional elements joined in series while the right one is the Maxwell branch (spring and dashpot in series). This model is called simplified fractional model.

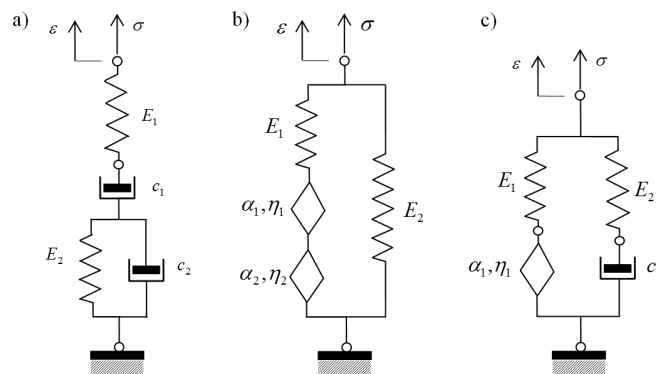


Fig. 2. Classical Burgers model (a), fractional Huet-Sayegh model (b) and simplified fractional model (c)

The problem of parameters' estimation of Burgers and Huet-Sayegh models based on experiments was considered in [8]. In case of the Burgers model the following results were obtained:  $E_1 = 12446$  MPa,  $E_2 = 7195$  MPa,  $c_1 = 368$  MPa  $\cdot$  s and  $c_2 = 126$  MPa  $\cdot$  s. For the Huet-Sayegh model the procedure of the curve fitting leads to the values as follows:  $E_1 = 1003 \cdot 10^5$  MPa,  $E_2 = 0$  MPa,  $\alpha_1 = 0.787$ ,  $\alpha_2 = 0.247$ ,  $\eta_1 = 660$  MPa  $\cdot$  s $^{\alpha_1}$ ,  $\eta_2 = 2985$  MPa  $\cdot$  s $^{\alpha_2}$ . It was proved in [8] that using the fractional model one can better fit the experiments within wide range of the excitation frequencies. Applying the curve fitting procedure for simplified fractional model gives the following results:  $E_1 = 12599$  MPa,  $E_2 = 1574 \cdot 10^4$  MPa,  $\alpha_1 = 0.640$ ,  $\eta_1 = 623$  MPa  $\cdot$  s $^{\alpha_1}$ ,  $c = 7.716$  MPa  $\cdot$  s.

In order to establish the characteristics of rheological models shown in Fig. 2, we use their transfer functions. In case of Burgers model we have the following equation

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$$E_B^*(s) = \frac{1}{\frac{1}{E_1} + \frac{1}{s c_1} + \frac{1}{E_2 + s c_2}} \quad (7)$$

The transfer functions in case of fractional Huet-Sayegh model  $E_H^*$  and fractional simplified model  $E_S^*$  are as follows

$$E_H^*(s) = E_2 + \frac{1}{\frac{1}{E_1} + \frac{1}{\eta_1 s^{\alpha_1}} + \frac{1}{\eta_2 s^{\alpha_2}}} \quad (8a)$$

$$E_S^*(s) = \frac{1}{\frac{1}{E_1} + \frac{1}{\eta_1 s^{\alpha_1}}} + \frac{1}{\frac{1}{E_2} + \frac{1}{s c}} \quad (8b)$$

For linear systems being analysed we can write the following equation relating stresses and strains

$$\sigma^*(s) = E^*(s) \varepsilon^*(s), \quad (9)$$

where  $\sigma^*(s)$  and  $\varepsilon^*(s)$  denote Laplace transforms of the stress  $\sigma(t)$  and strain  $\varepsilon(t)$  states respectively. Having analytical forms of the Laplace transforms for excitations suited for evaluation of such characteristics as creep/relaxations curves and hysteretic loops one can find the solution using inverse Laplace transform. This operation is carried out numerically using algorithms described in [9].

The results of numerical calculations are shown in Figs. 3–5. It shows the differences in characteristics for the classical Burgers model, fractional Huet-Sayegh model and simplified fractional model proposed herein.

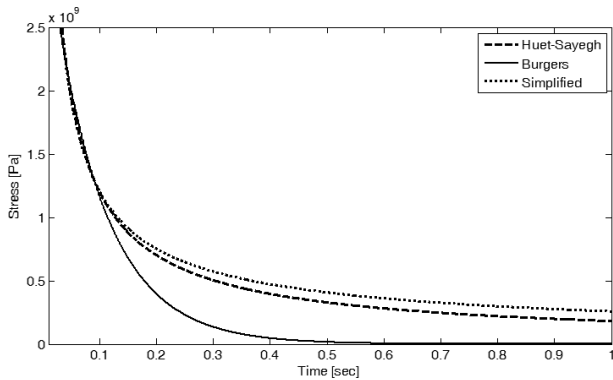


Fig. 3. Stress relaxation curves for Burgers and Huet-Sayegh models

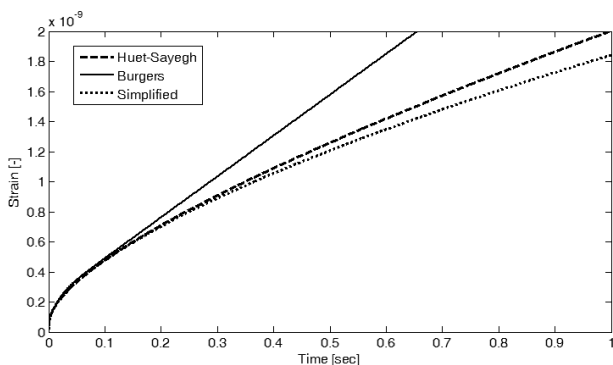


Fig. 4. Creep compliance behaviour of Burgers and Huet-Sayegh models

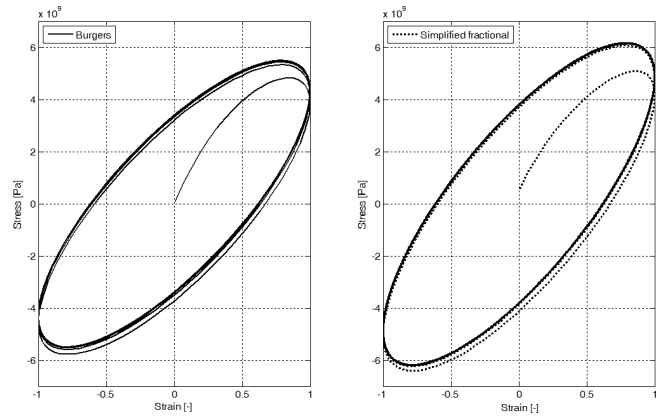


Fig. 5. Hysteretic loops of Burgers model and simplified fractional model (strain excitation)

The majority of methods used for numerical calculation of inverse Laplace transforms have serious limitations concerning the class of functions that can be inverted or the achievable accuracy. The procedures applied in this paper can be used for analysis of fractional rheological models. The required accuracy of the results can be enhanced without changing the algorithm, only at the cost of a longer computation time.

3. Viscoelastoplastic model

Typical creep and the recovery relation for asphaltic materials is shown in Fig. 6. There are three characteristic sections visualized in this graph. The section “a” is equal to immediate, time-independent strains and the section “b” equals the recoverable part of the strain. The “c” section is a permanent strain.

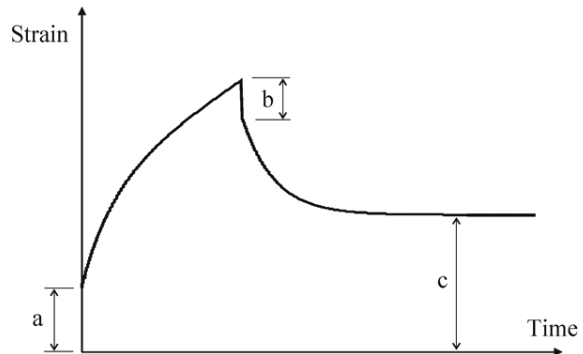


Fig. 6. Typical strain-time relation in creep and recovery test

In case of linear viscoelastic models, time-independent behaviour is always associated with elastic properties of the material while the permanent deformations are associated with viscosity phenomenon. The lengths of sections “a” and “b” are the same assuming viscoelastic behaviour of the material.

Viscoelastic rheological schemes shown in previous section can only model time-dependent permanent deformations associated with viscosity. Applying large stresses results in additional time-independent plastic deformations. As a consequence the lengths of the sections “a” and “b” visualized in Fig. 6 are not the same and obey the relation  $a > b$ . Modelling

both viscous and plastic permanent pavement deformations is a crucial issue in the process of rutting prediction. With the increase of traffic loads and tire pressures, most of the permanent deformation occurs in the upper layers of the road structure (viscoplastic rutting).

We propose a generalization of the Burgers model including plasticity phenomenon. The original rheological model is shown in Fig. 7. The additional elastoplastic network composed of the spring and slider in parallel is used. The limit stress in the slider modelling plasticity is denoted by  $\sigma_0$ .

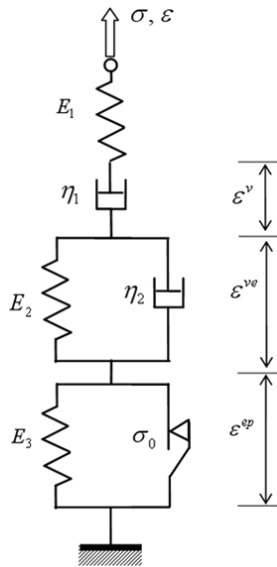


Fig. 7. Viscoelastoplastic rheological model of asphalt-aggregate mixture

The total deformation of the body is composed of four parts. The first one is time-independent elastic deformation of the spring  $E_1$ . The second part is viscous permanent deformation of the dashpot  $\eta_1$ . We introduced an internal variable  $\varepsilon^v$  in order to describe viscous strains. Another strain variable describing viscoelastic part of the strain is  $\varepsilon^{ve}$ . Finally the elastoplastic permanent deformations are modelled using the variable  $\varepsilon^{ep}$ .

The system of constitutive relationships of the proposed model can be formulated in the following form

$$\begin{aligned} \sigma &= E_1 (\varepsilon - \varepsilon^v - \varepsilon^{ve} - \varepsilon^{ep}), \\ \sigma^p &= \sigma - E_3 \varepsilon^{ep}, \\ \dot{\varepsilon}^v &= f^v(\sigma), \\ \dot{\varepsilon}^{ve} &= f^{ve}(\sigma, \varepsilon^{ve}), \\ \dot{\varepsilon}^{ep} &= f^{ep}(\sigma, \sigma^p, \varepsilon^{ve}, \dot{\varepsilon}), \end{aligned} \tag{10a}$$

where  $\sigma^p$  denotes plastic stress in the slider. The functions  $f^v$  and  $f^{ve}$  describing viscous and viscoelastic rates of deformation respectively, are easy to formulate

$$\begin{aligned} f^v(\sigma) &= \frac{1}{\eta_1} \sigma \\ f^{ve}(\sigma, \varepsilon^{ve}) &= \frac{1}{\eta_2} (\sigma - E_2 \varepsilon^{ve}). \end{aligned} \tag{10b}$$

The crucial problem is to write the elastoplastic strain rate function  $f^{ep}$ . The procedure of so called differential successions which may be used for this purpose has been described in [10–12]. Differential successions define additional relations which should be satisfied by the time derivatives of variables, describing the constitutive relation. The whole algorithm is complicated, thus we present only the final result

$$f^{ep}(\sigma, \sigma^p, \varepsilon^{ve}, \dot{\varepsilon}) = \begin{cases} 0 & \text{if } |\sigma^p| < \sigma_0, \\ \frac{E_1}{\sigma^p(E_1 + E_3)} \left[ \sigma^p \left( \dot{\varepsilon} + \frac{E_2}{\eta_2} \varepsilon^{ve} - \frac{\sigma}{\eta_{eq}} \right) \right]^+ & \text{if } |\sigma^p| = \sigma_0, \end{cases} \tag{10c}$$

where

$$\eta_{eq} := \frac{\eta_1 \eta_2}{\eta_1 + \eta_2}, \quad [z]^+ := \begin{cases} z & \text{if } z > 0, \\ 0 & \text{if } z \leq 0. \end{cases} \tag{10d}$$

Integrating the system of differential Eq. (10) we can evaluate the stress history based on the given strain excitations.

The viscoelastoplastic model presented in this Section depends on six material parameters. The values of viscoelastic components  $E_1, E_2, \eta_1$  and  $\eta_2$  can be obtained based on curve fitting procedure of the creep test, because the solution of the Burgers equation with static step excitation has an analytical form [4]. The results of such a procedure using algorithms implemented in MATLAB software are shown in Fig. 8 and in Table 1. The remaining parameters  $E_3$  and  $\sigma_0$  can be easily established observing the creep-recovery test results. If the applied stress  $\sigma$  is bigger than the plastic limit  $\sigma_0$ , then the following relations hold

$$a = \frac{\sigma - \sigma_0}{E_3} + b, \quad b = \frac{\sigma}{E_1}. \tag{11}$$

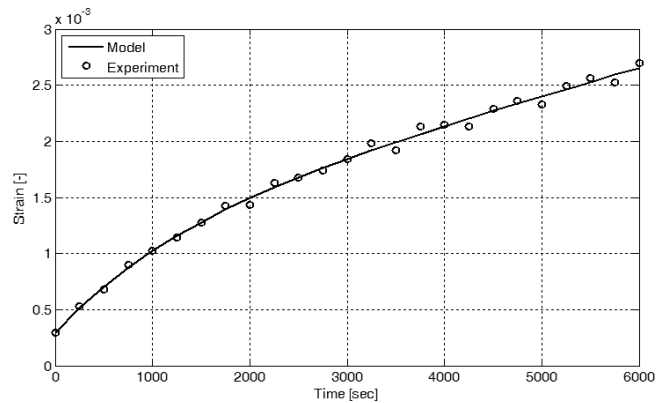


Fig. 8. Curve-fitting result for viscoelastic parameters' evaluation

Table 1  
Parameters of viscoelastoplastic model

$E_1$	$E_2$	$E_3$	$\eta_1$	$\eta_2$	$\sigma_0$
[MPa]	[MPa]	[MPa]	[GPa · s]	[GPa · s]	[MPa]
5130	1650	1540	6160	2150	1.0

Using Eqs. (11) along with Fig. 6 we can calculate the values of  $E_3$  and  $\sigma_0$  for certain material. The set of parameters obtained using the above mentioned method is present-

ed in Table 1. The experimental results were assumed based on [13].

Figure 9 presents numerical results of the creep and recovery simulations. Two models were considered taking the parameters shown in Table 1 – viscoelastic and viscoelasto-plastic. The amplitude of step excitation for creep behaviour modelling was equal to 2.0 MPa. The results show how the plastic limit stress influences the value of intermediate deformations.

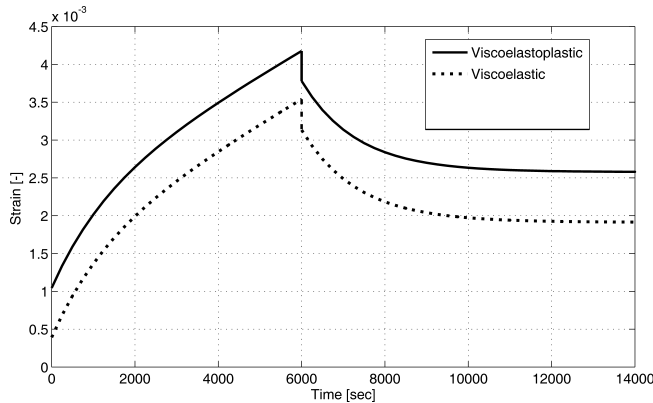


Fig. 9. Strain-time relation in creep and recovery numerical test for viscoelastic and viscoelastoplastic models

The idea of modelling elastic-visco-plastic properties of asphalt-aggregate mixes via rheological schemes is not new [14]. It should be noted, however, that the new approach proposed in this paper is related to a method which produces constitutive relationships in the form of explicit non-linear differential equations that are not presented in the literature. A differential form of constitutive equations allows straightforward implementation of integration algorithms.

#### 4. Three-dimensional constitutive equations

The stress tensor  $\sigma$  as well as the strain tensor  $\epsilon$ , can be decomposed additively into volumetric and deviatoric parts. The following relations hold

$$\sigma = \mathbf{p} + \mathbf{s} \quad \text{where} \quad \mathbf{s} := \sigma - \frac{1}{3} \text{tr}(\sigma) \mathbf{I}, \quad (12)$$

$$\epsilon = \mathbf{a} + \mathbf{e} \quad \text{where} \quad \mathbf{e} := \epsilon - \frac{1}{3} \text{tr}(\epsilon) \mathbf{I}. \quad (13)$$

In the above expressions the symbols  $\mathbf{p}$  and  $\mathbf{a}$  denote volumetric stress and strain tensor respectively, while  $\mathbf{s}$  and  $\mathbf{e}$  denote stress and strain deviators. Moreover,  $\mathbf{I}$  denotes  $2^{nd}$  rank identity tensor and the symbol “tr” denotes the trace operation.

This part of the study is devoted to the analysis of three-dimensional constitutive relationships suited for modelling of viscous and plastic permanent deformations of asphalt-aggregate mixes. The idea of the Burgers’ model generalization taking into account plasticity phenomenon is used. The investigations is focused on differential description of the proposed equations.

Let us begin with an assumption limiting the viscosity and plasticity phenomena to the deviatoric subspaces of stresses and strains. Thus, the volumetric deformations are elastic, recoverable and time-independent. For such a behaviour modelling, the yield function should be pressure independent. The simplest hypothesis we can use in this case is the Huber-Mises-Hencky (HMH) condition. We formulate constitutive equations taking into account rheological scheme shown in Fig. 10.

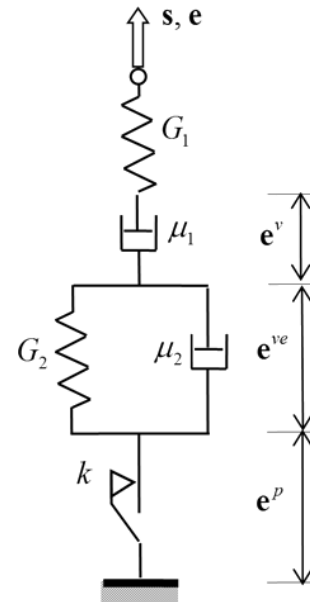


Fig. 10. Rheological schemes of viscoelastic-plastic model in deviatoric subspace

The constitutive relationships are as follows

$$\begin{aligned} \mathbf{s} &= 2G_1 (\mathbf{e} - \mathbf{e}^v - \mathbf{e}^{ve} - \mathbf{e}^p), \\ \dot{\mathbf{e}}^v &= f_B^v(\mathbf{s}), \\ \dot{\mathbf{e}}^{ve} &= f_B^{ve}(\mathbf{s}, \mathbf{e}^{ve}), \\ \dot{\mathbf{e}}^p &= f_B^p(\mathbf{s}, \mathbf{e}^{ve}, \dot{\mathbf{e}}). \end{aligned} \quad (14a)$$

The mappings describing non-elastic strain rates can be expressed in the following form

$$\begin{aligned} f_B^v(\mathbf{s}) &= \frac{1}{2\mu_1} \mathbf{s}, \\ f_B^{ve}(\mathbf{s}, \mathbf{e}^{ve}) &= \frac{1}{2\mu_2} (\mathbf{s} - 2G_2 \mathbf{e}^{ve}), \\ f_B^p(\mathbf{s}, \mathbf{e}^{ve}, \dot{\mathbf{e}}) &= \begin{cases} 0 & \text{if } \|\mathbf{s}\| < \sqrt{2}k, \\ \lambda \mathbf{s} & \text{if } \|\mathbf{s}\| = \sqrt{2}k, \end{cases} \end{aligned} \quad (14b)$$

where

$$\begin{aligned} \lambda &= \frac{1}{2k^2} \left[ \mathbf{s} \cdot \dot{\mathbf{e}} + \frac{G_2}{\mu_2} \mathbf{s} \cdot \mathbf{e}^{ve} - \frac{k^2}{\mu_{eq}} \right]^+ \\ \text{and} \quad \mu_{eq} &:= \frac{\mu_1 \mu_2}{\mu_1 + \mu_2}. \end{aligned} \quad (14c)$$

Equations (14) should be completed by a formula  $\mathbf{p} = 3K\mathbf{a}$ , describing volumetric behaviour where  $K$  denotes bulk modulus.

The proposed model expressed via Eq. (14) is relatively simple but has a limiting possibility for modelling of asphalt-aggregate mixes. First of all let us note that the yield condition taken seems to be too simple in case of asphaltic materials. Next model we formulate is more complicated and it uses the idea of pressure-dependency. Thus, a yield condition depends on volumetric stresses and exhibits hardening. We still use the assumption limiting viscosity to the deviatoric subspaces of strains and stresses.

The decomposition rule assumed for stresses and strains can be visualized based on rheological schemes shown in Fig. 11. The tensor of deviatoric strains  $\mathbf{e}$  is decomposed onto elastic, plastic  $\mathbf{e}^p$ , viscous  $\mathbf{e}^v$  and viscoelastic  $\mathbf{e}^{ve}$  parts (see Fig. 11a). Both elastic and viscoelastic strains are recoverable, while the plastic and the viscous strains are permanent. On the other hand, the intensity of the elastic and the plastic strains depends only on the intensity of external loading. The values of viscoelastic and viscous strains depend additionally on the loading rates. We use additional kinematic variables describing deviatoric plastic stress state  $\mathbf{s}^p$  and volumetric plastic stress state  $\mathbf{p}^p$ .

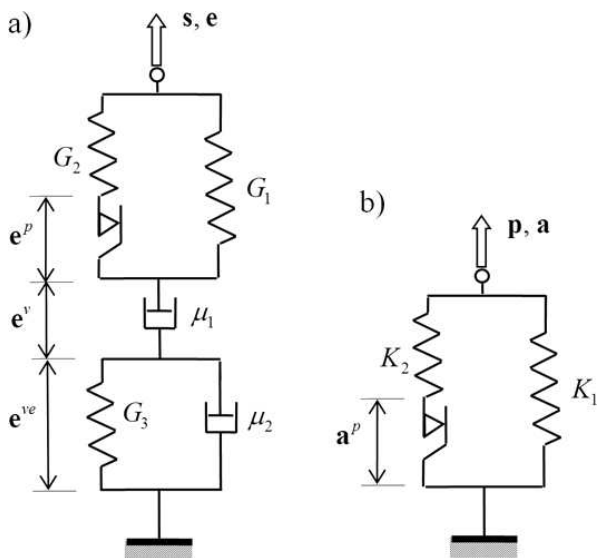


Fig. 11. Rheological schemes of asphalt-aggregate model in deviatoric (a) and volumetric (b) subspaces

In case of volumetric strains (see Fig. 11b), the elastic-plastic model with kinematic hardening was assumed. The volumetric plastic strains are denoted by  $\mathbf{a}^p$ .

Elastic constants  $K_1$  and  $K_2$  shown in Fig. 11b denote bulk moduli. The constants  $G_1$ ,  $G_2$  and  $G_3$  (Fig. 11a) denote shear moduli while  $\mu_1$  and  $\mu_2$  describe deviatoric viscosity. The plastic properties of the model depend on a yield condition. In the paper, a pressure dependent Mises-Schleicher (MS) hypothesis was assumed [13, 15]. The MS yield function can be expressed in the following form

$$F^{MS}(\sigma^p) := \|\mathbf{s}^p\|^n + \alpha R^{n-1} \text{tr} \sigma^p - R^n, \quad (15a)$$

where

$$\begin{aligned} R &= \sqrt{2}k_{eq}, \\ k_{eq} &= \frac{G_2}{G_1 + G_2}k, \\ \sigma^p &= \mathbf{s}^p + \mathbf{p}^p, \\ \mathbf{s}^p &= 2G_2(\mathbf{e} - \mathbf{e}^v - \mathbf{e}^{ve} - \mathbf{e}^p), \\ \mathbf{p}^p &= 3K_2(\mathbf{a} - \mathbf{a}^p). \end{aligned} \quad (15b)$$

In Eq. (15) additional material constants  $k$ ,  $\alpha$  and  $n$  are introduced. The  $k$  denotes pure shear plastic limit. The equivalent plastic limit  $k_{eq}$  is also used, because within an experiment the total stresses are measured, while the yield function expressed via Eq. (15a) describes only plastic stresses  $\sigma^p$ . The form of Eq. (15b) defining  $k_{eq}$  results from the rheological scheme modelling kinematic hardening.

Dimensionless parameters  $\alpha \geq 0$  and  $n \geq 1$  describe the shape of the yield function in meridional plane (Fig. 12). In deviatoric plane, the MS function is of circular shape with variable radius depending on the value of  $\text{tr} \sigma^p$ . Assuming  $\alpha = 0$  in Eq. (15a) leads to the HMM yield function. Taking  $n = 1$  in Eq. (15a) we obtain the Drucker-Prager (DP) condition.

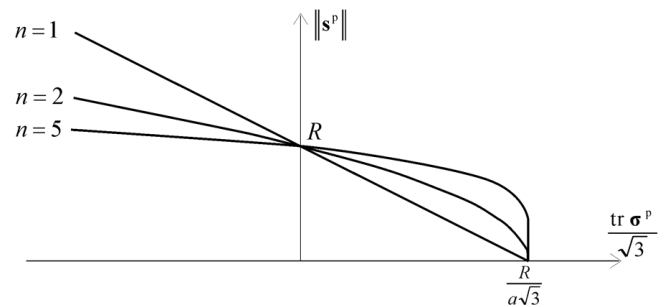


Fig. 12. MS yield functions in meridional plane for selected values of parameter "n"

The proposed model can be reduced to the classical viscoelastic Burgers scheme taking  $k \rightarrow \infty$ . Assuming  $\mu_1 \rightarrow \infty$  and  $\mu_2 \rightarrow \infty$  the model exhibits only elastic-plastic properties with kinematic hardening. Such an interpretation allows to determine the constants  $G_1$  and  $G_2$  based on pure shear test. The bulk moduli  $K_1$  and  $K_2$  can be determined based on the values of Young modulus  $E$  and Poisson ratio  $\nu$ . The remaining parameters should be evaluated applying creep tests within the range of small loadings not provoking any plastic behaviour.

The system of constitutive equations has complicated form. The total stress state is decomposed into volumetric and deviatoric parts (see Eq. (12a)). The volumetric part can be described as follows

$$\begin{aligned} \mathbf{p} &= 3K_1\mathbf{a} + 3K_2(\mathbf{a} - \mathbf{a}^p), \\ \dot{\mathbf{a}}^p &= f_{vol}^p(\sigma, \mathbf{e}^{ve}, \dot{\mathbf{e}}), \end{aligned} \quad (16a)$$

where

$$f_{vol}^p(\sigma, \mathbf{e}^{ve}, \dot{\mathbf{e}}) = \begin{cases} \mathbf{0} & \text{if } F^{MS}(\sigma^p) < 0, \\ \lambda \alpha R^{n-1} \mathbf{I} & \text{if } F^{MS}(\sigma^p) = 0. \end{cases} \quad (16b)$$

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The deviatoric stresses are described by the following relationships

$$\begin{aligned} \mathbf{s} &= 2G_1(\mathbf{e} - \mathbf{e}^v - \mathbf{e}^{ve}) + \mathbf{s}^p, \\ \dot{\mathbf{e}}^v &= f_{dev}^v(\mathbf{s}), \\ \dot{\mathbf{e}}^{ve} &= f_{dev}^{ve}(\mathbf{s}, \mathbf{e}^{ve}), \\ \dot{\mathbf{e}}^p &= f_{dev}^p(\boldsymbol{\sigma}, \mathbf{e}^{ve}, \dot{\mathbf{e}}), \end{aligned} \quad (16c)$$

taking into account the following equations defining the rates of internal variables

$$\begin{aligned} f_{dev}^v(\mathbf{s}) &= \frac{1}{2\mu_1}\mathbf{s}, \\ f_{dev}^{ve}(\mathbf{s}, \mathbf{e}^{ve}) &= \frac{1}{2\mu_2}(\mathbf{s} - 2G_3\mathbf{e}^{ve}), \\ f_{dev}^p(\boldsymbol{\sigma}, \mathbf{e}^{ve}, \dot{\mathbf{e}}) &= \begin{cases} \mathbf{0} & \text{if } F^{MS}(\boldsymbol{\sigma}^p) < 0, \\ \lambda n \|\mathbf{s}^p\|^{n-2} \mathbf{s}^p & \text{if } F^{MS}(\boldsymbol{\sigma}^p) = 0. \end{cases} \end{aligned} \quad (16d)$$

The Lagrange multiplier  $\lambda$  (see Eq. (16b) and Eq. (16d)) has an analytical explicit form

$$\lambda = \frac{\left[ 2G_2 n \|\mathbf{s}^p\|^{n-2} \mathbf{s}^p \left( \dot{\mathbf{e}} - \frac{1}{2\mu_{eq}} \mathbf{s} + \frac{G_3}{\mu_2} \mathbf{e}^{ve} \right) + 3K_2 \alpha R^{n-1} \text{tr} \dot{\mathbf{e}} \right]^+}{2G_2 n^2 \|\mathbf{s}^p\|^{2(n-1)} + 9K_2 \alpha^2 R^{2(n-1)}}. \quad (16e)$$

It may be proved that taking  $K_2 = 0$ ,  $G_1 = 0$ ,  $\mu_1 \rightarrow \infty$ ,  $\mu_2 \rightarrow \infty$ ,  $n = 2$  and  $\alpha = 0$  one can obtain the constitutive relationships of the elastic-perfectly plastic material obeying HMH yield hypothesis.

The system of constitutive equations (16) has a differential explicit form in case of the MS yield condition. Analysing more complex yield criteria with non-circular deviatoric sections it may be impossible to obtain an analytical formula for the Lagrange multiplier  $\lambda$  (Eq. (16e)). In such a case the value of  $\lambda$  should be obtained numerically within of each step of integration of constitutive relationships.

## 5. Conclusions

In the paper we presented the characteristics of viscoelastic rheological models of asphalt-aggregate mixes. Interesting results were obtained in Sec. 2 using simplified fractional model, which can be an alternative to Huet-Sayegh (1965) model. It should be emphasized that the simplified model we proposed has only 5 parameters which has to be calculated using curve-fitting procedure. It is an advantage with respect to 6-parameter Huet-Sayegh model.

One-dimensional viscoelastoplastic model as well as its three-dimensional generalization presented in Sec. 3 and Sec. 4 respectively were obtained based on the procedure explained in [10]. Such material models may be used for constitutive description of multilayer road structure subjected to dynamic or quasi-static loads [16, 17].

The main advantage of the method being proposed here-in is that the differential equations defining the behaviour of

asphalt-aggregate mixes are of explicit type. Thus, the existence of the solution as well as its uniqueness can be proved. On the other hand, it allows straightforward implementations in the FEM commercial codes.

In case of 3D models the choice of an appropriate yield criterion should be justified via experimental tests which are not standard in case of asphalt-aggregate mixes [18]. Additional testing is required over the static and dynamic range of load rates in order to evaluate plastic and viscous properties of the material. For example, somewhat similar material model presented in [19] was calibrated based on direct tensile test results carried out for various strain rates. In our case such an experiment should be completed by testing the material in pure shear stress state because of the complexity of the yield surface.

There are number of possible enhancements to the current 3D model. For instance, the assumption limiting the viscosity phenomenon to deviatoric subspace should be revised. On the other hand, in the case of large pressure-stress states, the material obeying Mises-Schleicher yield condition does not exhibit plasticity phenomenon. Thus, some modification of yield function leading to so called cap models should also be considered [20]. Another problem needed to be considered is a generalization of viscoelastic models described by fractional derivatives including plasticity phenomenon. For example generalization of the Huet-Sayegh model taking into account permanent plastic deformations should be proposed. The work in this area is currently underway by the author.

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