BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 61, No. 2, 2013
*DOI: 10.2478/bpasts-2013-0031*

# Controllability of dynamical systems. A survey

J. KLAMKA*

Institute of Control Engineering, Silesian University of Technology, 16 Akademicka St., 44-100 Gliwice, Poland

**Abstract.** The main objective of this article is to review the major progress that has been made on controllability of dynamical systems over the past number of years. Controllability is one of the fundamental concepts in the mathematical control theory. This is a qualitative property of dynamical control systems and is of particular importance in control theory. A systematic study of controllability was started at the beginning of sixties in the last century, when the theory of controllability based on the description in the form of state space for both time-invariant and time-varying linear control systems was worked out.

Roughly speaking, controllability generally means, that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. It should be mentioned, that in the literature there are many different definitions of controllability, which strongly depend on a class of dynamical control systems and on the other hand on the form of admissible controls.

Controllability problems for different types of dynamical systems require the application of numerous mathematical concepts and methods taken directly from differential geometry, functional analysis, topology, matrix analysis and theory of ordinary and partial differential equations and theory of difference equations. In the paper we use mainly state-space models of dynamical systems, which provide a robust and universal method for studying controllability of various classes of systems.

Controllability plays an essential role in the development of modern mathematical control theory. There are various important relationships between controllability, stability and stabilizability of linear both finite-dimensional and infinite-dimensional control systems. Controllability is also strongly related to the theory of realization and so called minimal realization and canonical forms for linear time-invariant control systems such as the Kalmam canonical form, the Jordan canonical form or the Luenberger canonical form. It should be mentioned, that for many dynamical systems there exists a formal duality between the concepts of controllability and observability. Moreover, controllability is strongly connected with the minimum energy control problem for many classes of linear finite dimensional, infinite dimensional dynamical systems, and delayed systems both deterministic and stochastic.

Finally, it is well known, that controllability concept has many important applications not only in control theory and systems theory, but also in such areas as industrial and chemical process control, reactor control, control of electric bulk power systems, aerospce engineering and recently in quantum systems theory.

**Key words:** controllability, dynamical systems, control theory.

## 1. Introduction

Control theory is an interdisciplinary branch of engineering and mathematics that deals with influence behavior of dynamical systems. Controllability is one of the fundamental

concepts in mathematical control theory. This is a qualitative property of dynamical control systems and it is of particular importance in control theory. Systematic study of controllability was started at the beginning of sixties in the last century, when the theory of controllability based on the description in the form of state space for both time-invariant and time-varying linear control systems was worked out.

Roughly speaking, controllability generally means, that it is possible to steer dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. It should be mentioned, that in the literature there are many different definitions of controllability, which strongly depend on one hand on a class of dynamical control systems and on the other hand on the form of admissible controls.

In recent years various controllability problems for different types of linear semilinear and nonlinear dynamical systems have been considered in many publications and monographs. Moreover, it should be stressed, that the most literature in this direction has been mainly concerned with different controllability problems for dynamical systems with unconstrained controls and without delays in the state variables or in the controls.

The main purpose of the paper is to present without mathematical proofs a review of recent controllability problems for a wide class of dynamical systems. Moreover, it should be pointed out, that exact mathematical descriptions of controllability criteria can be found for example in the following publications [1–29].

## 2. Controllability significance

Controllability plays an essential role in the development of modern mathematical control theory. There are various important relationships between controllability, stability and stabilizability of linear both finite-dimensional and infinite-dimensional control systems. Controllability is also strongly related with the theory of realization and so called minimal realization and canonical forms for linear time-invariant con-

*e-mail: jerzy.klamka@polsl.pl

trol systems such as the Kalmam canonical form, the Jordan canonical form or the Luenberger canonical form. It should be mentioned, that for many dynamical systems there exists a formal duality between the concepts of controllability and observability. Moreover, controllability is strongly connected with the minimum energy control problem for many classes of linear finite dimensional, infinite dimensional dynamical systems, and delayed systems both deterministic and stochastic.

Therefore, controllability criteria are useful in the following branches of mathematical control theory:

- stabilizability conditions, canonical forms, minimum energy control and minimal realization for positive systems,
- stabilizability conditions, canonical forms, minimum energy control and minimal realization for fractional systems,
- minimum energy control problem for a wide class of stochastic systems with delays in control and state variables,
- duality theorems, canonical forms and minimum energy control for infinite dimensional systems,
- controllability, duality, stabilizability, mathematical modeling and optimal control of quantum systems.

Controllability has many important applications not only in control theory and systems theory, but also in such areas as industrial and chemical process control, reactor control, control of electric bulk power systems, aerospce engineering and recently in quantum systems theory.

Systematic study of controllability was started at the beginning of the sixties in the 20-th century, when the theory of controllability based on the description in the form of state space for both time-invariant and time-varying linear control systems was worked out. The extensive list of these publications can be found for example in the monographs [10] and [11] or in the survey papers [12] and [22].

During last few years quantum dynamical systems have been discussed in many publications. This fact is motivated by possible applications in the theory of quantum informatics [30–33]. Quantum control systems are either defined in finite-dimensional complex space or in the space of linear operators over finite-dimensional complex space. In the first case the quantum states are called state vectors and in the second density operators.

Control system description of a quantum closed system is described by bilinear ordinary differential state equation in the form of Schrödinger equation for state vectors and Liouville [34, 35] equation for density matrices. Therefore, controllability investigations require using special mathematical methods as Lie groups and Lie algebras.

Traditional controllability concept can be extended for so called structural controllability, which may be more reasonable in case of uncertainties [10, 11]. It should be pointed out, that in practice most of system parameter values are difficult to identify and are known only to certain approximations. Thus structural controllability, which is independent of a specific value of unknown parameters are of particular interest. Roughly speaking, linear system is said to be structurally controllable if one can find a set of values for the free parameters such that the corresponding system is controllable in the standard sense [10, 11].

Structural controllability of linear control system is strongly related to numerical computations of distance from a given controllable switched linear control system to the nearest an uncontrollable one [10, 11].

First of all let us observe, that from algebraic characterization of controllability and structural controllability immediately follows that controllability is a generic property in the space of matrices defining such systems [10, 11]. Therefore, the set of controllable switched systems is an open and dense subset. Hence, it is important to know how far a controllable linear system is from the nearest uncontrollable linear system. This is especially important for linear systems with matrices whose coefficients are given with some parameter uncertainty.

An explicit bound for the distance between a controllable linear control system to the closed set of uncontrollable switched linear control system can be obtained using special norm defined for the set of matrices and singular value decomposition for controllability matrix [10, 11].

## 3. Nonlinear and semilinear dynamical systems

The last decades have seen a continually growing interest in controllability theory of dynamical systems. This is clearly related to the wide variety of theoretical results and possible applications. Up to the present time the problem of controllability for continuous-time and discrete-time linear dynamical systems has been extensively investigated in many papers (see e.g. [10–12, 36] for extensive list of references). However, this is not true for the nonlinear dynamical systems especially with different types of delays in control and state variables, and for nonlinear dynamical systems with constrained controls.

Similarly, only a few papers concern constrained controllability problems for continuous or discrete semi-linear dynamical systems. It should be pointed out, that in the proofs of controllability results for nonlinear and semi-linear dynamical systems linearization methods and generalization of open mapping theorem [37–41] are extensively used. The special case of nonlinear dynamical systems are semi-linear systems. Let us recall that semi-linear dynamical control systems contain linear and pure nonlinear parts in the differential state equations [15, 37, 42, 43].

## 4. Infinite-dimensional systems

Infinite-dimensional dynamical control systems plays a very important role in mathematical control theory. This class consists of both continuous-time systems and discrete-time systems [10–12, 22 ,36]. Continuous-time infinite-dimensional systems include for example, a very wide class of so-called distributed parameter systems described by numerous types of partial differential equations defined in bounded or unbounded regions and with different boundary conditions.

For infinite-dimensional dynamical systems it is necessary to distinguish between the notions of approximate and exact controllability [10, 11]. It follows directly from the fact that in infinite-dimensional spaces there exist linear subspaces which are not closed. On the other hand, for nonlinear dynamical systems there exist two fundamental concepts of controllability; namely local controllability and global controllability [10, 11]. Therefore, for nonlinear abstract dynamical systems defined in infinite-dimensional spaces the following four main kinds of controllability are considered: local approximate controllability, global approximate controllability, local exact controllability and global exact controllability [10–12, 22].

Controllability problems for finite-dimensional nonlinear dynamical systems and stochastic dynamical systems have been considered in many publications; see e.g. [10, 11, 22, 26, and [27], for review of the literature. However, there exist only a few papers on controllability problems for infinite-dimensional nonlinear systems [42–49].

Among the fundamental theoretical results, used in the proofs of the main results for nonlinear or semi-linear dynamical systems, the most important include:

- generalized open mapping theorem,
- spectral theory of linear unbounded operators,
- linear semi-groups theory for bounded linear operators,
- Lie algebras and Lie groups,
- fixed-point theorems such as Banach, Schauder, Schaefer and Nussbaum theorems,
- theory of completely positive trace preserving maps,
- mild solutions of abstract differential and evolution equations in Hilbert and Banach spaces.

**4.1. Nonlinear neutral impulsive integrodifferential evolution systems in Banach spaces.** In various fields of science and engineering, many problems that are related to linear viscoelasticity, nonlinear elasticity and Newtonian or non-Newtonian fluid mechanics have mathematical models which are described by differential or integral equations or integrodifferential equations. This part of the paper centers around the controllability for dynamical systems described by the integrodifferential models. Such systems are modelled by abstract delay differential equations. In particular abstract neutral differential equations arise in many areas of applied mathematics and, for this reason, this type of equation has been receiving much attention in recent years and they depend on the delays of state and their derivative. Related works of this kind can be found in [44–52].

The study of differential equations with traditional initial value problem has been extended in several directions. One emerging direction is to consider the impulsive initial conditions. The impulsive initial conditions are combinations of traditional initial value problems and short-term perturbations, whose duration can be negligible in comparison with the duration of the process. Several authors [44–52] have investigated controllability of the impulsive differential equations.

As far as the controllability problems associated with finite-dimensional systems modelled by ordinary differential equations are concerned, this theory has been extensively studied during the last decades. In the finite-dimensional context, a system is controllable if and only if the algebraic Kalman rank condition is satisfied. According to this property, when a system is controllable for some time, it is controllable for all time. But this is no longer true in the context of infinite-dimensional systems modelled by partial differential equations.

The large class of scientific and engineering problems modelled by partial differential and integrodifferential equations can be expressed in various forms of differential and integrodifferential equations in abstract spaces. It is interesting to study the controllability problem for such models in Banach spaces. The controllability problem for first and second order nonlinear functional differential and integrodifferential systems in Banach spaces has been studied by many authors by using semigroup theory, cosine family of operators and various fixed point theorems for nonlinear operators [42] and [43] such as Banach theorem, Nussbaum theorem, Schaefer theorem, Schauder theorem, Monch theorem or Sadovski theorem.

In recent years, the theory of impulsive differential equations has provided a natural frame work for mathematical modelling of many real world phenomena, namely in control, biological and medical domains. In these models, the investigated simulating processes and phenomena are subjected to certain perturbations whose duration is negligible in comparison with the total duration of the process. Such perturbations can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses. These process tend to be more suitably modelled by impulsive differential equations, which allow for discontinuities in the evolution of the state.

On the other hand, the concept of controllability is of great importance in mathematical control theory. The problem of controllability is to show the existence of a control function, which steers the solution of the system from its initial state to final state, where the initial and final states may vary over the entire space. Many authors have studied the controllability of nonlinear systems with and without impulses, see for instance [7, 12, 14, 16, 37–38, 40, 43].

In recent years, significant progress has been made in the controllability of linear and nonlinear deterministic systems [39, 44, 45] and the nonlocal initial condition which in many cases, has much better effect in applications then the traditional initial condition. The nonlocal initial value problems can be more useful than the standard initial value problems to describe many physical phenomena of dynamical systems. It should be pointed out, that the study of Volterra-Fredholm integrodifferential equations plays an important role for abstract formulation of many initial, boundary value problems of perturbed differential partial integro-differential equations.

Recently, many authors have studied about mixed type integrodifferential systems without (or with) delay conditions. Moreover, controllability of impulsive functional differential systems with nonlocal conditions has been studied by using the measures of noncompactness and Monch fixed point theorem and some sufficient conditions for controllability have been established.

It should be mentioned, that without assuming the compactness of the evolution system the existence, uniqueness and continuous dependence of mild solutions for nonlinear mixed type integrodifferential equations with finite delay and nonlocal conditions has been also established.. The results were obtained by using Banach fixed point theorem and semi-group theory. More recently, the existence of mild solutions for the nonlinear mixed type integro-differential functional evolution equations with nonlocal conditions was derived and the results were achieved by using Monch fixed point theorem and fixed point theory.

To the best of our knowledge, up to now no work reported on controllability of impulsive mixed Volterra- Fredholm functional integrodifferential evolution differential system with a finite delay and nonlocal conditions.

**4.2. Second order impulsive functional integrodifferential systems in Banach spaces.** Second order differential equations arise in many areas of science and technology whenever a relationship involving some continuously changing quantities and their rates of change are known. In particular, second order differential and integrodifferential equations serve as an abstract formulation of many partial integrodifferential equations which arise in problems connected with the transverse motion of an extensible beam, the vibration of hinged bars and many other physical phenomena. So it is quite significant to study the controllability problem for such systems in Banach spaces.

The concept of controllability involves the ability to move a system around in its entire configuration space using only certain admissible manipulations. The exact definition varies slightly within the framework of the type of models. In many cases, it is advantageous to treat the second order abstract differential equations directly rather than to convert them to first order systems. In the proofs of controllability criteria some basic ideas from the theory of cosine families of operators, which is related to the second order equations are often used.

Damping may be mathematically modelled as a force synchronous with the velocity of the object but opposite in direction to it. The occurrence of damped second order equations can be found in [44] and [45]. The branch of modern applied analysis known as "impulsive" differential equations furnishes a natural framework to mathematically describe some jumping processes.

The theory of impulsive integrodifferential equations and their applications to the field of physics have formed a very active research topic since the theory provides a natural framework for mathematical modelling of many physical phenomena [38] and [42]. In spite of the great possibilities for applications, the theory of these equations has been developing rather slowly due to obstacles of theoretical and technical character. The study of the properties of their solutions has been of an ever growing interest.

Recently, most efforts have been focused on the problem of controllability for various kinds of impulsive systems using different approaches [47] and [48]. In neutral delay differential equations, the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. Neutral differential equations arise in many fields and they depend on the delays of state and its derivative. Related works of this kind of equation can be found in [40] and [49]. For the fundamental solution of second order evolution system, one can refer the paper [50].

## 5. Stochastic systems

Classical control theory generally is based on deterministic approaches. However, uncertainty is a fundamental characteristic of many real dynamical systems. Theory of stochastic dynamical systems is now a well-established topic of research, which is still in intensive development and offers many open problems. Important fields of application are economics problems, decision problems, statistical physics, epidemiology, insurance mathematics, reliability theory, risk theory and others methods based on stochastic equations. Stochastic modelling has been widely used to model the phenomena arising in many branches of science and industry such as biology, economics, mechanics, electronics and telecommunications. The inclusion of random effects in differential equations leads to several distinct classes of stochastic equations, for which the solution processes have differentiable or non-differentiable sample paths. Therefore, stochastic differential equations and their controllability require many different method of analysis.

The general theory of stochastic differential equations both finite-dimensional and infinite-dimensional and their applications to the field of physics and technique can be found in the many mathematical monographs and related papers. This theory formed a very active research topic since provides a natural framework for mathematical modelling of many physical phenomena.

Controllability, both for linear or nonlinear stochastic dynamical systems, has recently received the attention of many researchers and has been discussed in several papers and monographs, in which where many different sufficient or necessary and sufficient conditions for stochastic controllability were formulated and proved [19, 26, 27, 53, 54]. However, it should be pointed out that all these results were obtained only for unconstrained admissible controls, finite dimensional state space and without delays in state or control.

Stochastic controllability problems for stochastic infinite-dimensional semi-linear impulsive integrodifferential dynamical systems with additive noise and with or without multiple time-varying point delays in the state variables are also discussed in the literature. The proofs of the main results are based on certain theorems taken from the theory of stochastic processes, linearization methods for stochastic dynamical systems, theory of semi-groups of linear operators, different fixed-point theorems as Banach, Schauder, Schaefer, or Nussbaum fixed-point theorems and on so-called generalized open mapping theorem presented and proved in the survey paper [45–54].

## 6. Delayed systems

Up to the present time the problem of controllability in continuous and discrete time linear dynamical systems has been extensively investigated in many papers (see e.g. [10–12, 16, 19, 20, 55]). However, this is not true for the nonlinear or semi-linear dynamical systems, especially with delays in control and with constrained controls. Only a few papers concern constrained controllability problems for continuous or discrete nonlinear or semi-linear dynamical systems with constrained controls [20, 23].

Dynamical systems with distributed [7] delays in control and state variable were also considered. Using some mapping theorems taken from functional analysis and linear approximation methods sufficient conditions for constrained relative and absolute controllability will be derived and proved.

Let us recall that semi-linear dynamical control systems with delays may contain different types of delays, both in pure linear and pure nonlinear parts, in the differential state equations. Sufficient conditions for constrained local relative controllability near the origin in a prescribed finite time interval for semi-linear dynamical systems with multiple variable point delays or distributed delays in the control and in the state variables, which nonlinear term is continuously differentiable near the origin are presented in [20] and [23].

In the above mentioned papers it is generally assumed that the values of admissible controls are in a given convex and closed cone with vertex at zero, or in a cone with nonempty interior. The proof of the main result are based on a so called generalized open mapping theorem presented in the paper [41]. Moreover, necessary and sufficient conditions for constrained global relative controllability of an associated linear dynamical system with multiple point delays in control are also discussed.

## 7. Positive systems

In recent years, the theory of positive dynamical systems has become a natural frame work for mathematical modelling of many real world phenomena, namely in control, biological and medical domains. Positive dynamical systems are of fundamental importance to numerous applications in different areas of science such as economics, biology, sociology and communication.

Positive dynamical systems both linear and nonlinear are dynamical systems with states, controls and outputs belonging to positive cones in linear spaces. Therefore, in fact positive dynamical systems are nonlinear systems. Among many important developments in control theory over last two decades, control theory of positive dynamical systems [55] has played an essential role.

Controllability, reachability and realization problems for finite dimensional positive both continuous-time and discrete-time dynamical systems were discussed for example in monograph [55] and paper [25], using the results taken directly from the nonlinear functional analysis and especially from the theory of semi-groups of bounded operators and general theory of unbounded linear operators.

## 8. Fractional systems

The development of controllability theory both for continuous-time and discrete-time dynamical systems with fractional derivatives and fractional difference operators has seen considerable advances since the publication of papers [56–59] and monograph [60]. Although classic mathematical models are still very useful, large dynamical systems prompt the search for more refined mathematical models, which leads to better understanding and approximations of real processes.

The general theory of fractional differential equations and fractional impulsive integrodifferential equations and their applications to the field of physics and technique can be found in the monograph [60]. This theory formed a very active research topic since provides a natural framework for mathematical modelling of many physical phenomena. In particular, the fast development of this theory has allowed to solve a wide range of problems in mathematical modelling and simulation of certain kinds of dynamical systems in physics and electronics. Fractional derivative techniques provide useful exploratory tools, including the suggestion of new mathematical models and the validation of existing ones.

Mathematical fundamentals of fractional calculus and fractional differential and difference equations are given in the monographs [60], and in the related papers [55-59]. Most of the earliest work on controllability for fractional dynamical systems was related to linear continuous-time or discrete-time systems with limited applications of the real dynamical systems. In addition, the earliest theoretical work concerned time-invariant processes without delays in state variables or in control.

Using the results presented for linear fractional systems and applying linearization method the sufficient conditions for local controllability near the origin are formulated and proved in the paper [25]. Moreover, applying generalized open mapping theorem in Banach spaces [41] and linear semi-group theory in the paper [36] the sufficient conditions for approximation controllability in finite time with conically constrained admissible controls are formulated and proved.

## 9. Quantum dynamical systems

Fast recent development of quantum information field in both theory and experiments caused increased interest in new methods of quantum systems control. Various models for open-loop and closed-loop control scenarios for quantum systems have been developed in recent years [30–35].

Quantum systems can be classified according to their interaction with the environment. If a quantum system exchange neither information nor energy with its environment it is called closed and its time evolution is described completely by a Hamiltonian and its respective unitary operator. On the other hand if the exchange of information or energy occurs, the system is called open.

Due to the destructive nature of quantum measurement in some models one has to be constrained to open-loop control of a quantum system. This fact means that during the time evolution of the quantum system it is physically impossible to extract any information about the state of the system.

In the simplest case open-loop control of the closed quantum system is described by the bi-linear model. In this case the differential equation of the evolution is described by the sum of the drift Hamiltonian and the control Hamiltonians. The parameters of the control Hamiltonians may be constrained in various ways due to physical constraints of the system. Many quantum systems can be only controlled locally, which means that control Hamiltonians act only on one of the Hilbert spaces that constitute larger tensor product Hilbert space of the system.

The control constrained to local operations is of a great interest in various applications, especially in quantum computation and spin graph systems. Other possible constraints, such as constrained energy or constrained frequency, are possible. They are very important in the scope of optimal control of quantum systems.

In the most generic case open quantum systems are not controllable with coherent, unitary control due to the fact that the action of the generic completely positive trace preserving map cannot be reversed unitarily. For example Markovian dynamics of finite-dimensional open quantum system is not coherently controllable. However, many schemes of incoherent control of open quantum systems have been described. Some of these schemes are based on the technique known as quantum error correcting codes. In incoherent control schemes quantum unitary evolution together with quantum measurements is used to drive the system to the desired state even if quantum noise is present in the system.

## 10. Switched systems

The last decades have seen a continually growing interest in controllability theory of hybrid dynamical systems and their special case named switched dynamical systems. In the literature there have been a lot of papers for controllability both continuous-time and discrete-time switched systems [61–71]. Switched systems deserve investigation for theoretical interest as well as for practical applications. Switching system structure is an essential feature of many engineering control applications such as power systems and power electronics. From a theoretical point of view switched linear system consists of several linear subsystems and a rule that organize switching among them.

Hybrid systems which are capable of exhibiting simultaneously several kinds of dynamic behavior in different parts of the system (e.g., continuous-time dynamics, discrete-time dynamics, jump phenomena, logic commands) are of great current interest (see, e.g., [61, 64, 67]). Examples of such systems include the Multiple-Models, Switching and Tuning paradigm from adaptive control, Hybrid Control Systems, and a plethora of techniques that arise in Event Driven Systems are typical examples of such systems of varying degrees of complexity. Moreover, hybrid systems include computer disk drives, transmission an stepper motors, constrained robotic systems, intelligent vehicle/highway systems, sampled-data systems, discrete event systems, and many other types of dynamical systems.

Switched linear systems are hybrid systems that consist of several linear subsystems and a rule of switching among them. Switched linear systems provide a framework which bridges the linear systems and the complex and/or uncertain systems. On one hand, switching among linear systems may produce complex system behaviors such as chaos and multiple limit cycles. On the other hand, switched linear systems are relatively easy to handle as many powerful tools from linear and multi-linear analysis are available to cope with these systems.

Moreover, the study of switched linear systems provides additional insights into some long-standing and sophisticated problems, such as intelligent control, adaptive control, and robust analysis and control. Theoretical examination of switched linear systems are academically more challenging due to their rich, diverse, and complex dynamics. Switching makes those systems much more complicated than standard-time invariant or even time-varying systems. Many more complicated behaviors/dynamics and fundamentally new properties, which standard systems do not have, have been demonstrated on switched linear systems. From the point of view of control system design, switching brings an additional degree of freedom in control system design. Switching laws, in addition to control laws, may be utilized to manipulate switched systems to achieve a better performance of a system. This can be seen as an added advantage for control design to attain certain control purposes like stabilizability or controllability.

For the controllability analysis of switched linear control systems, a much more difficult situation arises since both the control input and the switching rule are design variables to be determined. Thus, the interaction between them is very important from controllability point of view. Moreover, it should be mentioned that for switched linear discrete-time control system in general case the controllable set is not a subspace but a countable union of subspaces. For switched linear continuous-time control system, in general case the controllable set is an uncountable union of subspaces. Controllability results from different types of switched systems which can be found in [61-71].

## 11. Concluding remarks

Controllability problems for different types of dynamical systems require the application of numerous mathematical concepts and methods taken directly from differential geometry, functional analysis, topology, matrix analysis and theory of ordinary and partial differential equations and theory of difference equations. The state-space models of dynamical systems provides a robust and universal method for studying controllability of various classes of systems.

Finally, it should be stressed, that there are numerous open problems for controllability concepts for special types of dynamical systems. For example, it should be pointed out, that up to present time the most literature on controllability problems has been mainly concerned with unconstrained controls and without delays in the state variables or in the controls.

**Acknowledgements.** This paper was supported by the National Science Center under the grant no. DEC-2012/-

05/B/ST7/00065.

## REFERENCES

[1] J. Klamka, "Relative controllability and minimum energy control of linear systems with distributed delays in control", *IEEE Trans. on. Automatic Control* AC-21 (4), 594–595 (1976).
[2] J. Klamka, "Controllability of linear systems with time-variable delays in control", *Int. J. Control* 24 (6), 869–878 (1976).
[3] J. Klamka, "On the controllability of linear systems with delays in the control", *Int. J. Control* 25 (6), 875–883 (1977).
[4] J. Klamka, "Absolute controllability of linear systems with time-variable delays in control", *Int. J. Control* 26 (1), 57–63 (1977).
[5] J. Klamka, "Relative and absolute controllability of discrete systems with delays in control", *Int. J. Control* 26 (1), 65–74 (1977).
[6] J. Klamka, "Minimum energy control of discrete systems with delays in control", *Int. J. Control* 26 (5), 737–744 (1977).
[7] J. Klamka, "Relative controllability of nonlinear systems with distributed delays in control", *Int. J. Control* 28 (2), 307–312 (1978).
[8] J. Klamka, "Relative controllability of infinite-dimensional systems with delays in control", *Systems Science* 4 (1), 43–52 (1978).
[9] J. Klamka, "Minimum energy control of 2D systems in Hilbert spaces", *Systems Science,* 9 (1–2), 33–42 (1983).
[10] J. Klamka, *Controllability of Dynamical Systems*, PWN, Warszawa, 1990, (in Polish).
[11] J. Klamka, *Controllability of Dynamical Systems*, Kluwer Academic, Dordrecht, 1991.
[12] J. Klamka, "Controllability of dynamical systems – a survey", *Archives of Control Sciences* 2 (3/4), 281–307 (1993).
[13] J. Klamka, "Constrained controllability of nonlinear systems", *J. Math. Analysis and Applications* 201 (2), 365–374 (1996).
[14] J. Klamka, "Schauder's fixed point theorem in nonlinear controllability problems", *Control and Cybernetics* 29 (3), 377–393 (2000).
[15] J. Klamka, "Constrained exact controllability of semilinear systems", *Systems and Control Letters* 47 (2), 139–147 (2002).
[16] J. Klamka, "Constrained controllability of semilinear systems with multiple delays in control", *Bull. Pol. Ac.: Tech.* 52 (1), 25–30 (2004).
[17] J. Klamka, "Approximate constrained controllability of mechanical systems", *J. Theoretical and Applied Mechanics* 43 (3), 539–554 (2005).
[18] J. Klamka, "Controllability of dynamical systems", *Applied Mathematics* 9, 57–75 (2008).
[19] J. Klamka, "Stochastic controllability of systems with variable delay in control", *Bull. Pol. Ac.: Tech.* 56 (3), 279–284 (2008).
[20] J. Klamka, "Constrained controllability of semilinear systems with delayed controls", *Bull. Pol. Ac.: Tech.* 56 (4), 333–337 (2008).
[21] J. Klamka, "Stochastic controllability and minimum energy control of systems with multiple delays in control", *Applied Mathematics and Computation* 206 (2), 704–715 (2008).
[22] J. Klamka, "Controllability of fractional discrete-time systems with delay", *Scientific Notebooks of Silesian University of Technology. Series of Automatics* 151, 67–72 (2008).
[23] J. Klamka, "Constrained controllability of semilinear systems with delays" *Nonlinear Dynamics* 56 (1–2), 169–177 (2009).
[24] J. Klamka, "Stochastic controllability of systems with multiple delays in control", *Int. J. Applied Mathematics and Computer Science* 19 (1), 39–47 (2009).
[25] J. Klamka, "Controllability and minimum energy control problem of fractional discrete-time systems", in: *New Trends in Nanotechnology and Fractional Calculus*, eds: D. Baleanu, Z.B. Guvenc, and J.A. Tenreiro Machado, pp. 503–509, Springer-Verlag, New York, 2010.
[26] J. Klamka and L. Socha, "Some remarks about stochastic controllability", *IEEE Trans. on Automatic Control* AC-22 (5), 880–881 (1977).
[27] J. Klamka and L. Socha, "Some remarks about stochastic controllability for delayed linear systems", *Int. J. Control.* 32 (3), CD-ROM (1980).
[28] J. Klamka and J. Wyrwał, "Controllability of second-order infinite-dimensional systems, *Systems and Control Letters* 57 (5), 386–391 (2008).
[29] J. Klamka and B. Sikora, "On constrained stochastic controllability of dynamical systems with multiple delays in control, *Bull. Pol. Ac.: Tech.* 60 (2), 301–306 (2012).
[30] P. Gawron, J. Klamka, J. Miszczak, and R. Winiarczyk, "Extending scientific computing system with structural quantum programming capabilities", *Bull. Pol. Ac.: Tech.* 58 (1), 77–88 (2010).
[31] S. Węgrzyn, J. Klamka, and S. Bugajski, "Foundation of quantum computing, Part 1", *Archives of Theoretical and Applied Informatics* 13 (2), 97–142 (2001).
[32] S. Węgrzyn J. Klamka, and S. Bugajski, "Foundation of quantum computing. Part 2", *Archives of Theoretical and Applied Informatics* 14 (2), 93–106 (2002).
[33] S. Węgrzyn, J. Klamka, L. Znamirowski, R. Winiarczyk, and S. Nowak, "Nano and quantum systems of informatics", *Bull. Pol. Ac.: Tech.* 52 (1), 1–10 (2004).
[34] H.P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems,* Oxford University Press, Oxford, 2002.
[35] D. Dong and I.R. Petersen, "Quantum control theory and applications: a survey", *IET Control Theory and Applications* 4 (12), 2651–2671 (2010).
[36] N.K. Son, "A unified approach to constrained approximate controllability for the heat equations and the retarded equations", *J. Mathematical Analysis and Applications* 150 (1), 1–19 (1990).
[37] K. Balachandran and J.P. Dauer, "Controllability of nonlinear systems via fixed point theorems, *J. Optimization Theory and Applications* 53, 345–352 (1987).
[38] K. Balachandran and J.P. Dauer, "Controllability of nonlinear systems in Banach spaces. A Survey", *J. Optimization Theory and Applications* 115, 7–28 (2002).
[39] K. Balachandran, D.G. Park, and P. Manimegalai, "Controllability of second order integrodifferential evolution systems in Banach spaces", *Computers and Mathematics with Applications* 49, 1623–1642 (2005).
[40] K. Naito, "Controllability of semilinear control systems dominated by the linear part", *SIAM J. on Control and Optimization* 25 (3), 715–722 (1987).
[41] S.M. Robinson, "Stability theory for systems of inequalities. Part II. Differentiable nonlinear systems", *SIAM J. on Numerical Analysis* 13 (4), 497–513 (1976).
[42] G. Peichl and W. Schappacher, "Constrained controllability in Banach spaces", *SIAM J. on Control and Optimization* 24 (6), 1261–1275 (1986).
[43] T.I. Seidman, "Invariance of the reachable set under nonlinear

perturbations”, *SIAM J. on Control and Optimization* 25 (5), 1173–1191 (1987).

[44] J.Y. Park, K. Balachandran, and G. Arthi, “Controllability of impulsive neutral integrodifferential systems with infinite delay in Banach spaces”, *Nonlinear Analysis: Hybrid Systems* 3, 184–194 (2009).

[45] Y. Lin Y. and N. Tanaka, “Nonlinear abstract wave equations with strong damping”, *J. Mathematical Analysis and Applications* 225, 46–61 (1998).

[46] H.X. Zhou, “Controllability properties of linear and semilinear abstract control systems”, *SIAM J. on Control and Optimization* 22 (3), 405–422 (1984).

[47] X. Fu, “Controllability of abstract neutral functional differential systems with unbounded delay”, *Applied Mathematics and Computation* 141, 299–314 (2004).

[48] E. Hernandez and H.R. Henriquez, “Impulsive partial neutral differential equations”, *Applied Mathematical Letters* 19, 215–222 (2006).

[49] E. Hernandez, H.R. Henriquez, and M. Rabello, “Existence of solutions for impulsive partial neutral functional differential equations”, *J. Mathematical Analysis and Applications* 331, 1135–1158 (2007).

[50] X. Liu and A.R. Wilms, “Impulsive controllability of linear dynamical systems with applications to maneuvers of spacecraft”, *Mathematical Problems in Eng.* 2, 277–299 (1996).

[51] J.Y. Park, K. Balachandran, and G. Arthi, “Controllability of impulsive neutral integrodifferential systems with infinite delay in Banach spaces”, *Nonlinear Analysis: Hybrid Systems* 3, 184–194 (2009).

[52] R. Sakthivel, N.I. Mahmudov, and J.H. Kim, “On controllability of second-order nonlinear impulsive differential systems”, *Nonlinear Analysis* 71, 45–52 (2009).

[53] N.I. Mahmudov and A. Denker, “On controllability of linear stochastic systems”, *Int. J. Control* 73 (2), 144–151 (2000).

[54] N.I. Mahmudov and S. Zorlu, “Controllability of nonlinear stochastic systems”, *Int. J. Control* 76 (2), 95–104 (2003).

[55] T. Kaczorek, *Positive 1D and 2D Systems*, Springer-Verlag, London, 2002.

[56] T. Kaczorek, “Reachability and controllability to zero of cone fractional linear systems”, *Archives of Control Sciences* 17 (3), 357–367 (2007).

[57] T. Kaczorek, “Fractional positive continuous-time linear systems and their reachability”, *Int. J. Applied Mathematical Computation Science* 18 (2), 223–228 (2008).

[58] T. Kaczorek, “Positive fractional continuous time linear systems with singular pencils”, *Bull. Pol. Ac.: Tech.* 60 (1), 9–12 (2012).

[59] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[60] S. Ostalczyk, *Outline of Integral/Differential Calculus of Fractional Orders. Theory and Applications in Authomatics*, Łódź University of Technology Publishing House, Łódź, 2008, (in Polish).

[61] S.S. Ge, Z. Sun, and T.H. Lee, “Reachability and controllability of switched linear discrete-time system”, *IEEE Trans. on Automatic Control* AC-46 (9), 1437–1441 (2001).

[62] Z. Sun, S.S. Ge, and T.H. Lee, “Controllability and reachability criteria for switched linear system”, *Automatica* 38 (5), 775–786 (2002).

[63] Z. Sun and S.S. Ge, *Switched Linear Systems – Control and Design*, Springer, New York, 2004.

[64] Z. Sun and D.Z. Zheng, “On reachability and stabilization of switched linear control systems”, *IEEE Trans. on Automatic Control* AC-46 (2), 291–295 (2001).

[65] X. Liu, H. Lin and B.M. Chen, “Structural controllability of switched linear systems”, *ArXiv.1106.1703v2 [cs.SY]* (2012).

[66] G. Xie and L. Wang, “Controllability and stabilizability of switched linear systems”, *Systems and Control Letters* 48 (2), 135–155 (2003).

[67] Z. Sun and S.S. Ge, *Stability Theory of Switched Dynamical Systems*, Springer-Verlag, London, 2011.

[68] J. Clotet, J. Ferrer, and M.D. Magret, “Distance from a controllable switched linear system to an uncontrollable one”, $2^{nd}$ *Meeting on Linear Algebra Matrix Analysis and Applications* 1, CD-ROM (2010).

[69] Y. Qiao and D. Cheng, “On partitioned controllability of switched linear systems”, *Automatica* 45 (1), 225–229 (2009).

[70] T. Kaczorek, “Positive fractional 2D continuous discrete time linear systems”, *Bull. Pol. Ac.: Tech.* 59 (4), 575–579 (2011).

[71] T. Kaczorek, “Positive switched 2D linear systems described by the Roesser models”, *Eur. J. Control* 18 (3), 239–246 (2012).