

## SENSITIVITY ANALYSIS OF BUCKLING LOADS OF BISYMMETRIC I-SECTION COLUMNS WITH BRACING ELEMENTS

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The first order variation of critical loads of thin-walled columns with bisymmetric open cross-section due to some variations of the stiffness and location of bracing elements is derived. The considerations are based on the classical linear theory of thin-walled beams with non-deformable cross-section introduced by Vlasov [1]. Both lateral braces and braces that restraint warping and torsion of the cross-section have been taken into account. In the numerical examples dealing with I-column, the functions describing the influence of location of the braces with unit stiffness on the critical load of torsional and flexural buckling are derived. The linear approximation of the exact relation of the critical load due to the variation of the stiffness and location of braces is determined.

*Key words:* Sensitivity analysis, thin-walled beam, stiffener, critical load.

## 1. INTRODUCTION

Various structural elements as beams, columns or frames are made of thin-walled members. In real structures those elements are often connected to purlins or wall rails supporting corrugated sheeting, and are parts of bracing systems providing stability to the whole structure. All those elements may be considered as braces modelled as elastic restraints of columns and frames. Thin-walled members are often subjected to axial loads and therefore it is important to determine critical loads of those elements. The critical load depends not only on the geometrical dimensions or material characteristics of the column but also on the stiffness and position of the braces, that may be used in the construction. In the paper buckling loads of thin-walled column braced by different means of restraining elements are considered. Three kinds of elastic restraints are taken into account: the lateral supports, the warping prevention braces and the torsional restraints braces. As a warping brace of the column one can consider elements that connect flanges and reduce warping of the cross-section. Warping prevention restraints in a form of transverse stiffeners, longitudinal edge stiffeners or box stiffeners of stiffness  $k_{\Theta}$ , are presented in Fig. 1. Behaviour of those stiffeners and its effect on the column performance was analysed by PLUM and SVENSSON [2] and SZYMCAK

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*et al.* [3] among others. As torsional braces one can assume bridging members, which provide the lateral side support and undergo bending when column twists. Lateral braces prevent side displacements of the column at the brace points. Such restraining members of stiffness  $k_{\Theta}, k_v$  are also presented in Fig. 1. All above mentioned braces lead to an increase in the critical loads. There are many research activities related to requirements of bracing stiffness in order to stabilize structural elements. The influence of bracing stiffness on the buckling load of I-columns has been analysed by Gosowski [4] and by Gil and Yura [5], Waszczyszyn *et al.* [6], Weiss and Gizejowski [7] or Trahair [8], among others. Studies conducted by Girgin *et al.* [9] have shown that simplified formulae for the buckling length of frame compression members, present in design codes may yield erroneous results. In the case of steel trusses similar problem was analysed by Iwicki [10]. The minimal stiffness of braces that allows to consider full bracing condition of compressed member has been derived by Winter [11] and Yura [12], among others.

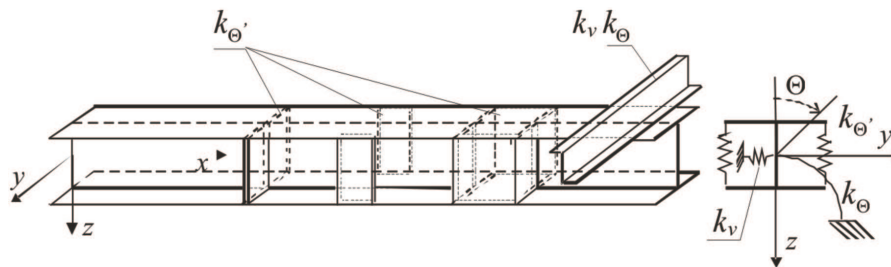


Fig. 1. Thin-walled column with out-of-plane multiple-restraints.  
Rys. 1. Dwuteowy pręt cienkościenny z różnymi usztywnieniami

In the present paper sensitivity analysis method has been used in research of critical loads of I-column with stiffeners. Sensitivity analysis was first developed by Haug *et al.* [13]. This method was successfully used by Szymczak [14] in research on the variation of critical load of I-section column due to variation of cross-section dimensions and material characteristics, as well as by Iwicki [15] in the case of residual stress variation and its influence on the critical load.

In the present work, the sensitivity analysis method is used to predict changes in the buckling load of columns as a result in the variation of brace localisation and stiffness. It is worth noting that in design code procedures, as for example in code PN-90/B-03200, the coefficient of buckling length is required and therefore both the flexural and torsional buckling loads have to be calculated. All kind of braces affect the buckling load level and the coefficient of buckling length so that the designer needs a tool for predicting the points where braces should be applied in order to efficiently increase the buckling load. The investigated problem is devoted to the analysis of the first order variation of critical loads of I-section column due to variations of the stiffness and location of bracing elements. The sensitivity analysis method is used to

determine the influence lines of location of the braces with unit stiffness on the critical buckling load of column. The linear approximation of the exact relation of the critical load due to the variation of the stiffness and location of braces is determined. The considerations are based on the classical linear theory of thin-walled members with non-deformable cross-section. Braces are modeled as linear elastic side supports. In the investigated examples braces are located at the center of the gravity of the cross-section so the torsion and bending of the column is considered independently.

## 2. FIRST VARIATION OF THE COLUMN BUCKLING LOAD DUE TO THE VARIATION OF BRACE STIFFNESS

### 2.1. FIRST VARIATION OF FLEXURAL-TORSIONAL BUCKLING LOAD

At first a column with continuously distributed restraints shown in Fig. 2a is considered. Lateral braces of stiffness  $k_v$  are situated at distance  $z_t$  from the centroid.

The total potential energy of the column can be written in following form (WEISS and GIŻEJOWSKI [7] or TRAHAIR [8]):

$$(2.1) \quad V = \frac{1}{2} \int_0^l (EJ_y w'^2 + EJ_z v'^2 + EJ_\omega \Theta'^2 + GJ_d \Theta'^2) dx +$$

$$\frac{1}{2} \int_0^l (k_v v^2 + k_\theta \Theta^2 + k_{\theta'} \Theta'^2 - 2k_v \Theta v (z_t - z_0) + k_v \Theta^2 (z_t - z_0)^2) dx$$

$$- \frac{1}{2} P_{cr} \int_0^l (v'^2 + w'^2 + \left(\frac{J_0}{A} + z_0^2\right) \Theta'^2 + 2z_0 v' \Theta') dx.$$

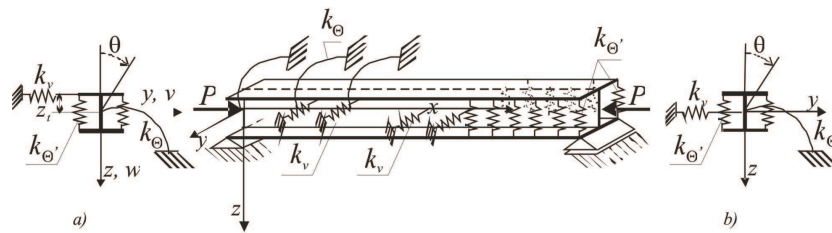


Fig. 2. Axially compressed I-section column with continuously distributed restraints: a) lateral brace is shifted from shear centre b) lateral brace is at the centroid.

Rys. 2. Pręt dwuteowy obciążony osiowo z usztywnieniami rozłożonymi w sposób ciągły:  
a) usztywnienia poprzeczne są położone poza środkiem ścinania, b) usztywnienia poprzeczne są położone w środku ciężkości przekroju

where:  $E$  – Young's modulus,  $G$  – shearing modulus,  $A$  – cross-section area,  $J_y, J_z$ ,  $J_0, J_\omega, J_d$ , moments of inertia, polar moment of inertia, warping and torsion section constants,  $k_v, k_\Theta, k_{\Theta'}$  – stiffnesses of the continuously distributed restraints,  $z_0$  is coordinate of shear centre. The primes denote the differentiation with respect to coordinate  $x$ . The first order variation of above equation due of the change in design variable  $d$  can be written as:

$$\begin{aligned}
 \delta V = & \int_0^l \left( EJ_z v'' \delta v'' + EJ_y w'' \delta w'' + EJ_\omega \Theta'' \delta \Theta'' + GJ_d \Theta' \delta \Theta' \right) dx + \\
 & \int_0^l \left( k_v v \delta v - k_v (z_t - z_0) (v \delta \Theta + \Theta \delta v) + k_{\Theta'} \Theta' \delta \Theta' + k_\Theta \Theta \delta \Theta + k_v (z_t - z_0)^2 \Theta \delta \Theta \right) dx + \\
 & -P_{cr} \int_0^l \left( v' \delta v' + w' \delta w' + \left( \frac{J_0}{A} + z_0^2 \right) \Theta' \delta \Theta' + z_0 (v' \delta \Theta' + \delta v' \Theta') \right) dx + \\
 (2.2) \quad & \frac{1}{2} \int_0^l \left( (EJ_y)_{,d} v''^2 + (EJ_z)_{,d} w''^2 + (EJ_\omega)_{,d} \Theta''^2 + (GJ_d)_{,d} \Theta'^2 \right) \delta d dx + \\
 & \frac{1}{2} \int_0^l \left( k_{v,d} v^2 - 2(k_v(z_t - z_0))_{,d} v \Theta + k_{\Theta',d} \Theta'^2 + k_{\Theta,d} \Theta^2 + (k_v(z_t - z_0)^2)_{,d} \Theta^2 \right) \delta d dx \\
 & - \frac{1}{2} P_{cr} \int_0^l \left( \left( \frac{J_0}{A} + z_0^2 \right)_{,d} \Theta'^2 + 2z_{0,d} v' \Theta' \right) \delta d dx \\
 & - \frac{1}{2} \delta P_{cr} \int_0^l \left( v'^2 + w'^2 + \left( \frac{J_0}{A} + z_0^2 \right) \Theta'^2 + 2z_0 v' \Theta' \right) dx = 0.
 \end{aligned}$$

The variation of the total potential energy at buckling state vanishes, and the first three integrals of Eq.(2.2) are zero because of virtual work theorem. The first variation of critical load for flexural-torsional buckling due to the variation of the design variable  $d$  takes the following form:

$$\begin{aligned}
 (2.3) \quad \delta P_{cr} = & \frac{\int_0^l \left( \begin{aligned} & (EJ_z)_{,d} v''^2 + (EJ_y)_{,d} w''^2 + (EJ_\omega)_{,d} \Theta''^2 + (GJ_d)_{,d} \Theta'^2 \\ & - P_{cr} \left( \left( \frac{J_0}{A} + z_0^2 \right)_{,d} \Theta'^2 + 2z_{0,d} v' \Theta' \right) + \\ & k_{v,d} v^2 - 2(k_v(z_t - z_0))_{,d} v \Theta + k_{\Theta',d} \Theta'^2 + k_{\Theta,d} \Theta^2 + (k_v(z_t - z_0)^2)_{,d} \Theta^2 \end{aligned} \right) \delta d dx}{\int_0^l \left( v'^2 + w'^2 + \left( \frac{J_0}{A} + z_0^2 \right) \Theta'^2 + 2z_0 v' \Theta' \right) dx} = \\
 & = \int_0^l \Lambda_{Pcr,d} \delta d dx
 \end{aligned}$$

The under-integral functions  $\Lambda_{P_{cr,d}}$  describe the influence of a column design variable variation on the buckling load.

## 2.2. FIRST VARIATION OF TORSIONAL BUCKLING LOAD

Let us consider a column with continuously distributed restraints presented in Fig. 2b. Because of bisymmetry of the cross-section, bending and torsion are not coupled and the buckling modes can be considered independently. Column is axially loaded and stiffened by means of continuously distributed elastic restraints that affect torsion and warping of the cross-section.

The total potential energy of the column is:

$$(2.4) \quad V = \frac{1}{2} \int_0^l (EJ_\omega \Theta''^2 + GJ_d \Theta'^2 + k_\Theta \Theta'^2 + k_\Theta \Theta^2) dx - \frac{1}{2} P_{cr} \int_0^l \frac{J_0}{A} \Theta'^2 dx.$$

The first order variation of above equation due the change in stiffness of braces can be written as:

$$(2.5) \quad \delta V = \int_0^l (EJ_\omega \Theta'' \delta \Theta'' + GJ_d \Theta' \delta \Theta' + k_\Theta \Theta' \delta \Theta' + k_\Theta \Theta \delta \Theta) dx - P_{cr} \int_0^l \frac{J_0}{A} \Theta' \delta \Theta dx + \frac{1}{2} \int_0^l (\delta k_\Theta \Theta'^2 + \delta k_\Theta \Theta^2) dx - \frac{1}{2} \delta P_{cr} \int_0^l \frac{J_0}{A} \Theta'^2 dx = 0.$$

The variation of the total potential energy at buckling state vanishes, and the first two integrals of Eq.(2.5) are zero because of virtual work theorem. The first variation of critical load for torsional buckling takes the following form:

$$(2.6) \quad \delta P_{cr} = \frac{\int_0^l (\delta k_\Theta \Theta'^2 + \delta k_\Theta \Theta^2) dx}{\int_0^l \frac{J_0}{A} \Theta'^2 dx} = \int_0^l \Lambda_{k_\Theta'}(x) \delta k_\Theta' dx + \int_0^l \Lambda_{k_\Theta}(x) \delta k_\Theta dx.$$

The under-integral functions  $\Lambda_{P_{cr,k_\Theta}}, \Lambda_{P_{cr,k_\Theta}'}$  describe the influence of a column brace with unit stiffness on the torsional buckling load.

### 2.3. FIRST VARIATION OF FLEXURAL BUCKLING LOAD

Consider an axially loaded column with continuously distributed elastic lateral restraints. The first variation of critical load for flexural buckling can be derived in a similar way as for torsional buckling. It yields:

$$(2.7) \quad \delta P_{cr} = \frac{\int_0^l (\delta k_v v^2) dx}{\int_0^l v'^2 dx} = \int_0^l \Lambda_{k_v}(x) \delta k_v dx$$

### 2.4. SENSITIVITY ANALYSIS OF BUCKLING LOAD OF DISCRETE STRUCTURAL SYSTEM

The equilibrium equation for structural systems can be written as follows:

$$(2.8) \quad (\mathbf{K} - P\mathbf{K}_G) \mathbf{z} = \mathbf{0},$$

where  $\mathbf{K}$  is the initial stiffness matrix component,  $\mathbf{K}_G$  is initial stress stiffness matrix component, and  $\mathbf{z}$  denotes the nodal displacement vector. Initial stiffness and initial stress matrices for member undergoing torsion were derived by SZYM CZAK [17], WASZCZYSZYN *et al.* [6] or WEISS and GIŻEJOWSKI [7]. Equation for the first variation of critical load with respect to the variation of design variables was derived by IWICKI [15] in the following form:

$$(2.9) \quad \delta P_{cr} = P_{cr,\mathbf{u}} \delta \mathbf{u} = \mathbf{z}^T (\mathbf{K}_{,\mathbf{u}} - P\mathbf{K}_{G,\mathbf{u}}) \mathbf{z} \delta \mathbf{u} = \Lambda_{P_{cr,\mathbf{u}}} \delta \mathbf{u}.$$

The vector  $\Lambda_{P_{cr,\mathbf{u}}}$  describes the influence of the unit change in the design variable vector  $\mathbf{u}$  on the buckling load. The above equation may be used as a tool for predicting the influence of variation of elastic restraint stiffness on the of buckling load. The presented method allows to calculate approximate buckling loads for larger variation of stiffness of elastic restraints. It allows to determinate possible position for brace application that may increase the buckling load in the most effective way.

## 3. FIRST VARIATION OF THE BUCKLING LOAD DUE TO THE VARIATION OF RESTRAINT LOCATION

### 3.1. FIRST VARIATION OF THE TORSIONAL CRITICAL LOAD DUE TO THE VARIATION OF BRACE LOCATION

Let us consider the column with an in-span elastic restraint situated at the position  $x_0$  (Fig. 3). The variation  $\delta x_0$  is assumed to be the design variable. The brace divides

column into two parts. In each part of the column governing differential equation has the form well known from the literature [1]:

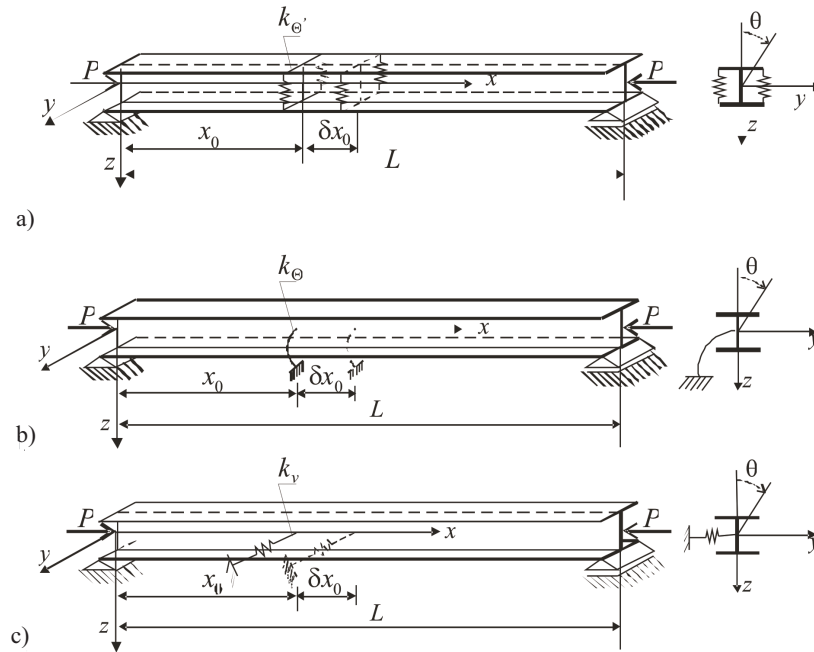


Fig. 3. Axially compressed I-section column with restraints of changed location: (a) warping, (b) torsional, and (c) lateral.

Rys. 3. Pręt dwuteowy obciążony osiowo z usztywnieniami, których położenie ulega zmianie: (a) usztywnienie ograniczające spłaszczenie, (b) skręcenie i usztywnienie poprzeczne (c)

$$(3.1) \quad (E_i J_{i\omega} \Theta_i'')'' + \left( \left( P_i \frac{J_{0i}}{A_i} - G_i J_{si} \right) \Theta_i' \right)' = 0$$

where index  $i = L, P$  denotes part of the column on the left and on the right sides of the brace position. After multiplying above equation by  $\lambda$  and integrating we obtain

$$(3.2) \quad 0 = \int_0^{l_1} \{G_L(\lambda_L, \Theta_L, \Theta_L', \Theta_L'')\} dx + \int_0^{l_2} \{G_P(\lambda_P, \Theta_P, \Theta_P', \Theta_P'')\} dx,$$

where  $\lambda$  is the Lagrange's multiplier,  $l_1, l_2$  are the lengths of parts of the column on the left and on the right side of the brace, and:

$$(3.3) \quad G_i(\lambda_i, \Theta_i, \Theta'_i, \Theta''_i) = \lambda_i \left[ (E_i J_{\omega i} \Theta''_i)'' + \left( \left( P_i \frac{J_{0i}}{A_i} - G J_{di} \right) \Theta'_i \right)' \right].$$

The increment of Eq.(3.2) due to change of brace location  $\delta x$  is:

$$(3.4) \quad \begin{aligned} 0 = & \int_0^{l_1 + \delta x} G_L(\lambda_L + \delta \lambda_L, \Theta_L + \delta \Theta_L, \Theta'_L + \delta \Theta'_L, \Theta''_L + \delta \Theta''_L) dx + \\ & + \int_0^{l_2} G_P(\lambda_P + \delta \lambda_P, \Theta_P + \delta \Theta_P, \Theta'_P + \delta \Theta'_P, \Theta''_P + \delta \Theta''_P) dx + \\ & - \int_0^{l_1} G_L(\lambda_L, \Theta_L, \Theta'_L, \Theta''_L) dx - \int_0^{l_2} G_P(\lambda_P, \Theta_P, \Theta'_P, \Theta''_P) dx. \end{aligned}$$

Expanding Eq.(3.4) into power series, and taking into account only the linear part of it, then after integration by parts one can obtain following relation for the first variation of torsional critical load:

$$\begin{aligned} \delta P_{cr} \int_0^l \lambda' \Theta' \frac{J_0}{A} = & \sum_{L,P} \int_0^l \left\{ \left[ -(E J_{\omega} \lambda'')'' - \left( \left( P \frac{J_0}{A} - G J_d \right) \lambda' \right)' + (k_{\Theta} \lambda')' - \lambda k_{\Theta} \right] \delta \Theta + \right. \\ & \left. - \left[ (E J_{\omega} \Theta'')'' + \left( \left( P \frac{J_0}{A} - G J_d \right) \Theta' \right)' - (k_{\Theta} \Theta')' + k_{\Theta} \Theta \right] \delta \lambda \right\} dx + \end{aligned}$$

$$(3.5) \quad \begin{aligned} & \left( -\lambda_L M'_{sL} + \lambda'_L B'_L + \bar{M}_{sL} \Theta'_L - \bar{B}_L \Theta''_L \right) \Big|_{l_1} \delta x - \left( -\lambda_P M'_{sP} + \lambda'_P B'_P + \bar{M}_{sP} \Theta'_P - \bar{B}_P \Theta''_P \right) \Big|_{l_1} \delta x + \\ & - \bar{M}_{sL} \Big|_{l_1} \delta \Theta_{Ll_1} + \bar{B}_L \Big|_{l_1} \delta \Theta'_{Ll_1} - \lambda'_L \Big|_{l_1} \delta B_{Ll_1} + \lambda_L \Big|_{l_1} \delta M_{sLl_1} + \\ & + \bar{M}_{sP} \Big|_{l_1} \delta \Theta_{Pl_1} - \bar{B}_P \Big|_{l_1} \delta \Theta'_{Pl_1} + \lambda'_P \Big|_{l_1} \delta B_{Pl_1} - \lambda_P \Big|_{l_1} \delta M_{sPl_1}. \end{aligned}$$

where the following relation for the boundary values of variations, and the following notation are used:



$$\begin{aligned}
 \delta M_s &= \delta \left( - (EJ_\omega \Theta'')' + \left( GJ_d - P \frac{J_0}{A} \right) \Theta' \right), \\
 \delta B &= -\delta (EJ_\omega \Theta''), \\
 \bar{B} &= -EJ_\omega \lambda'', \\
 \bar{M}_s &= - (EJ_\omega \lambda'')' + \left( GJ_d - P \frac{J_0}{A} \right) \lambda', \\
 (3.6) \quad \delta M_s|_0 &= \delta M_{s0} - M'_{s0} \delta x_0, \quad \delta M_s|_l = \delta M_{sl} - M'_{sl} \delta x_l, \\
 \delta B|_0 &= \delta B_0 - B'_0 \delta x_0, \quad \delta B|_l = \delta B_l - B'_l \delta x_l, \\
 \delta \Theta|_0 &= \delta \Theta_0 - \Theta'_0 \delta x_0, \quad \delta \Theta|_l = \delta \Theta_l - \Theta'_l \delta x_l, \\
 \delta \Theta'|_0 &= \delta \Theta'_0 - \Theta''_0 \delta x_0, \quad \delta \Theta'|_l = \delta \Theta'_l - \Theta''_l \delta x_l.
 \end{aligned}$$

Because the first variation of objective functional should be independent of displacement function and Lagrange's multiplier variations  $\delta\Theta$  and  $\delta\lambda$ , the under-integral part of Eq.(3.5) is zero. Under-integral parts of Eq. (3.5) represent differential equation of the primary and adjoint system. In the case of critical load variation, the primary and adjoint systems have the same buckling mode of the column. Taking into account natural boundary conditions in following form:

$$\begin{aligned}
 (3.7) \quad & - \bar{M}_{sL}|_{l_1} \delta \Theta_{Ll_1} + \bar{M}_{sP}|_{l_1} \delta \Theta_{Pl_1} = 0, \\
 & \bar{B}_L|_{l_1} \delta \Theta'_{Ll_1} - \bar{B}_P|_{l_1} \delta \Theta'_{Pl_1} = 0, \\
 & - \lambda'_L|_{l_1} \delta B_{Ll_1} + \lambda'_P|_{l_1} \delta B_{Pl_1} = 0, \\
 & \lambda_L|_{l_1} \delta M_{sLl_1} - \lambda_P|_{l_1} \delta M_{sPl_1} = 0,
 \end{aligned}$$

one can finally obtain the first variation of torsional critical load of the column due to a change of brace location:

$$(3.8) \quad \delta P_{cr} = \left\{ \begin{array}{l} (-\lambda_L M'_{sL} + \lambda'_L B'_L + \bar{M}_{sL} \Theta'_L - \bar{B}_L \Theta''_L) + \\ -(-\lambda_P M'_{sP} + \lambda'_P B'_P + \bar{M}_{sP} \Theta'_P - \bar{B}_P \Theta''_P) \end{array} \right\} \Big|_{l_1} \delta x \int_0^l r_0^2 \Theta'^2 dx.$$

### 3.1.1. First variation of the torsional critical load due to the variation of warping restraint location

Let us consider at first a column with braces of bimoment type (Fig. 3a). According to Plum and Svensson [2], the brace of bimoment type causes discontinuity of bimoment in the column. The warping stiffener that is connected to column flanges undergoes torsion enforced by warping of column flanges. In recent research Szymczak *et al.* [3] stiffeners have been modelled by means of shell elements and the stiffness of those elements was set with better accuracy. Aside from stiffener modelling, warping

restraining elements cause torsional angle of rotation and its first and third derivative, and also the torsional moment to be continuous in the cross-section with a stiffener of bimoment type applied. Thus:

$$(3.9) \quad \Delta B = k_{\Theta} \cdot \Theta' (x_0)$$

Taking into account continuity conditions, the first variation of torsional critical load due to the variation of warping preventing stiffener location  $\delta x_0$  takes the following form:

$$(3.10) \quad \delta P_{cr} = \left( -\bar{B}_L \Theta_L'' + \bar{B}_P \Theta_P'' \right) \Big|_{x_0} \delta x_0 \int_0^l r_0^2 \Theta'^2 dx = k_{\Theta} \cdot \Theta' (x_0) (\Theta_L'' + \Theta_P'') \delta x_0 \int_0^l r_0^2 \Theta'^2 dx.$$

### 3.1.2. First variation of the torsional buckling load due to the variation of torsional stiffener location

Let us consider now a column with torsional stiffeners (Fig. 3b). In the case of torsional stiffeners with the stiffness  $k_{\Theta}$ , the torsional angle and its first and second derivatives are continuous in the cross-section. Thus:

$$(3.11) \quad \Delta M (x_0) = k_{\Theta} \Theta (x_0),$$

Making use of Eq. (3.8), the first variation of torsional critical load due to the variation of torsional stiffener location  $\delta x_0$  can be written in the following form:

$$(3.12) \quad \delta P_{cr} (x_0) = \left( -\Delta M_{\omega} \lambda' - \Delta \bar{M}_{\omega} \Theta' \right) \Big|_{x_i} \delta x_0 \int_0^l r_0^2 \Theta'^2 dx = 2k_{\Theta} \Theta (x_0) \Theta' (x_0) \delta x_0 \int_0^l r_0^2 \Theta'^2 dx.$$

## 3.2. FIRST VARIATION OF THE FLEXURAL BUCKLING LOAD DUE TO THE VARIATION OF LATERAL BRACE LOCATION

Let us consider now a column with lateral braces (Fig. 3c). The deflection and its first and second derivatives are continuous in the cross-section. Thus:

$$(3.13) \quad \Delta T (x_0) = k_v v (x_0),$$

The first variation of flexural critical load due to the variation of lateral brace location  $\delta x_0$  can be expressed in the following form:

$$(3.14) \quad \delta P_{cr} = \left\{ (\lambda'_L M'_L + \bar{T}_{sL} v'_L) - (\lambda'_P M'_P + \bar{T}_P v'_P) \right\} \Big|_{x_0} \int_0^l v'^2 dx = 2k_v v (x_0) v' (x_0) \delta x_0 \int_0^l v'^2 dx.$$

#### 4. LINEAR APPROXIMATION OF EXACT RELATION BETWEEN CRITICAL LOAD AND STIFFNESS OR LOCATION OF BRACE

The first variation of critical load of column  $\delta P_{cr}$  due to the variation of the design variables may be used to determine the linear approximation of exact relation between critical buckling load of column and stiffness of braces in the following form:

$$(4.1) \quad P_{cr}(k) = P_{cr}(k_0) + \delta P_{cr}(\delta k = 1) \times (k - k_0).$$

where:  $k$  – actual brace stiffness,  $k_0$  – initial brace stiffness. The linear approximation of exact relation between critical buckling load of column and location of brace can be written in a similar form as for variation of bracing stiffness. It yields:

$$(4.2) \quad P_{cr}(k(x)) = P_{cr}(k(x_0)) + \delta P_{cr}(k(\delta x_0 = 1)) \times (x - x_0).$$

where:  $x$  – actual coordinate of brace that changes its position,  $x_0$  – initial coordinate of brace.

### 5. NUMERICAL EXAMPLES

#### 5.1. COLUMN WITH LATERAL BRACES

Let us consider, as a first example, the simply supported in both horizontal and vertical plane I-section column with two lateral braces placed at positions  $x = 1.6$  m and  $x = 3.6$  m (Fig. 4). The column is prevented from twisting at the supports, but warping of the cross-section is not prevented. Three variants of braces stiffness are analyzed. At first it is assumed that stiffness of lateral braces is  $k_{v_0} = 100$  kN/m, then this stiffness is set to be 500 kN/m and 1000 kN/m. The magnitude of assumed stiffnesses may be verified according design code formula. For example in the Polish design code PN-90/B-03200 the compressed element may be considered as side supported when the support is able to carry force equal to 1% of normal force in compressed part of the member, and when the lateral displacement is less than 1/200 of distance between braces. For the analyzed column the design normal force is about 600 kN. Assuming that the brace distance is  $l_0 = 4$  m the magnitude of approximated stiffness of lateral brace is:

$$(5.1) \quad k_v = \frac{0.01N}{l_0/200} = \frac{6kN}{0.02m} = 300kN/m.$$

The influence lines of variation of critical load of flexural buckling due to location of new stiffener of unit stiffness are presented in Fig. 5. Influence lines are related

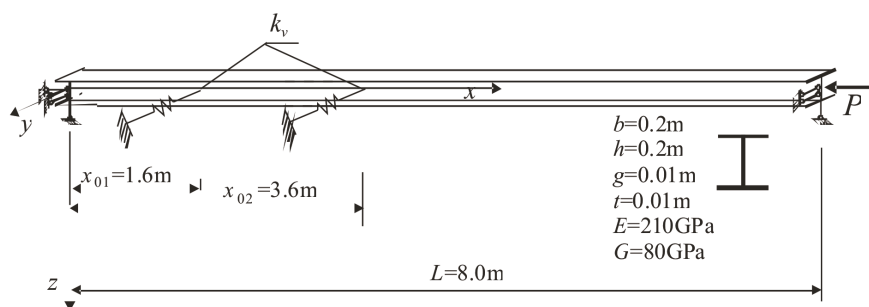


Fig. 4. Axially compressed thin-walled I-section column with two lateral restraints.  
Rys. 4. Pręt dwuteowy obciążony osiowo z dwoma usztywnieniami poprzecznymi

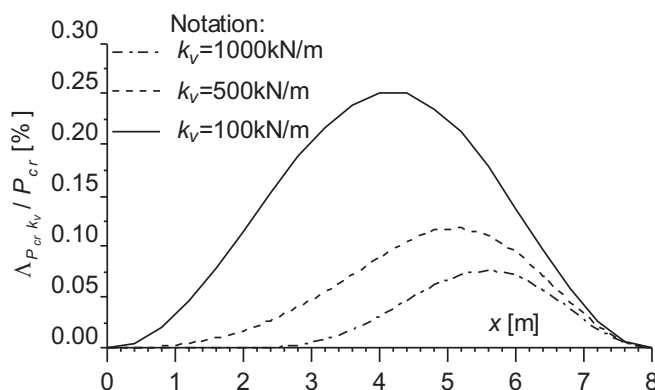


Fig. 5. The influence line of the variation of the flexural buckling load due to the location of the lateral restraint with unit stiffness for a column with two stiffeners at  $x = 1.6$  m and  $x = 3.6$  m of stiffness  $k_v = 100$  kN/m,  $k_v = 500$  kN/m,  $k_v = 1000$  kN/m.

Rys. 5. Linia wpływu względnej wariacji obciążenia krytycznego wyboczenia giętnego w zależności od dodania usztywnienia poprzecznego o jednostkowej sztywności dla pręta z dwoma usztywnieniami poprzecznymi o sztywności  $k_v = 100$  kN/m,  $k_v = 500$  kN/m,  $k_v = 1000$  kN/m usytuowanymi w przekrojach  $x = 1.6$  m i  $x = 3.6$  m

to the critical load of the column, namely:  $P_{cr0} = 642$  kN, 1300 kN and 1594 kN, respectively for three of analysed cases of the brace.

Then, the linear approximation of exact relation of the relative critical load of flexural buckling due to a change of the lateral brace stiffness for a column with stiffeners  $k_v = 100$  kN/m and with stiffeners  $k_v = 500$  kN/m was calculated. Lines are related to the critical load for column with the brace stiffness  $k_{v,0} = 100$  kN/m (Fig. 6). Linear approximations are determined according Eq.(4.1).

Next, the linear approximation of the relation of the relative critical load of the column due to a change of the brace location at  $x = 1.6$  m with the stiffness  $k_{\Theta} = 100$  kNm is drawn (Fig. 7). The approximation is determined according Eq.(4.2).

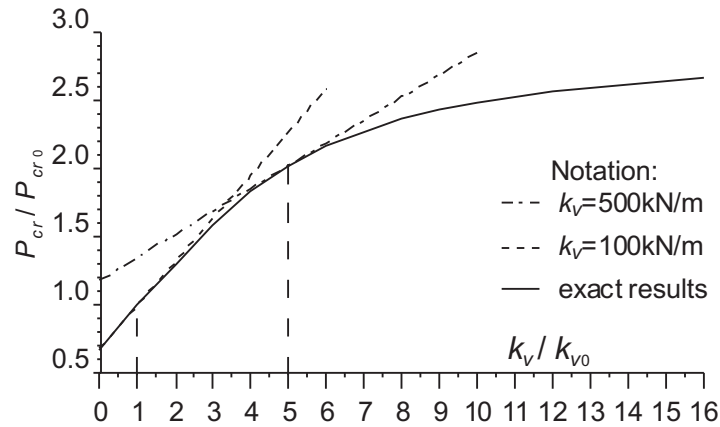


Fig. 6. The linear approximation of the exact relation of the relative flexural buckling load due to a change of the stiffness of lateral restraints at  $x=1.6$  m and  $x=3.6$  m for a column with stiffeners  $k_v = 100$  kN/m, and with stiffeners  $k_v = 500$  kN/m.

Rys. 6. Aproksymacja liniowa ścisłej zależności względnego obciążenia krytycznego wyboczenia giętnego w zależności od zmiany sztywności usztywnień poprzecznych o współrzędnych  $x=1.6$  m i  $x=3.6$  m, dla słupa z usztywnieniami o sztywności  $k_v = 100$  kN/m i  $k_v = 500$  kN/m

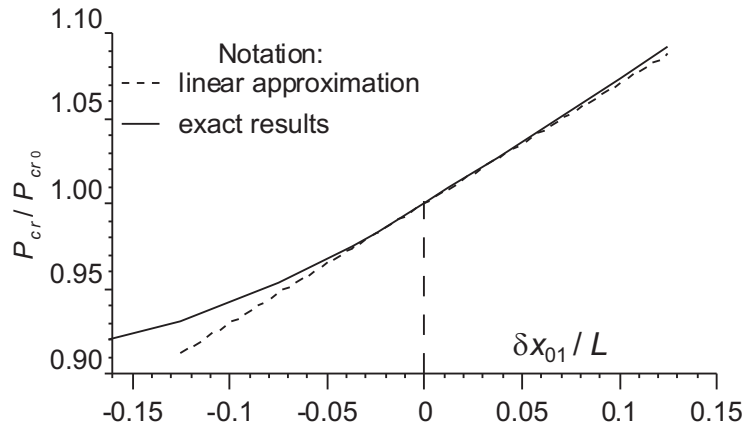


Fig. 7. The linear approximation of the exact relation of the relative flexural critical load of the column with two lateral restraints due to a change of the restraints location at  $x=1.6$  m of stiffness  $k_v = 100$  kN/m.

Rys. 7. Aproksymacja liniowa ścisłej zależności względnego obciążenia krytycznego wyboczenia giętnego słupa z dwoma usztywnieniami poprzecznymi w zależności od zmiany położenia usztywnienia o współrzędnej  $x=1.6$  m i o sztywności  $k_v = 100$  kN/m

## 5.2. COLUMN WITH WARPING STIFFENERS

The simply supported I-section column with two warping stiffeners placed in cross-sections at  $x = 1.6$  m and  $x = 3.6$  m is investigated (Fig. 8). It is assumed that the warping stiffness of stiffeners is  $k_{\theta_0} = 100$  kNm<sup>3</sup>. The range of stiffness of warping stiffener for the assumed I-section given in the Fig. 8 is determined by means of relations proposed by Gosowski [18]. Then two variants of column have been investigated: the column without warping stiffeners and with the stiffener stiffness equal to 100 kNm<sup>3</sup>. The critical load of column torsional buckling with stiffeners is  $P_{cr0} = 2868.5$  kN. The critical load of torsional buckling is 2047.2 kN for the column without stiffeners and 2604.6 kN for column with warping stiffeners of stiffness 50 kNm<sup>3</sup>.

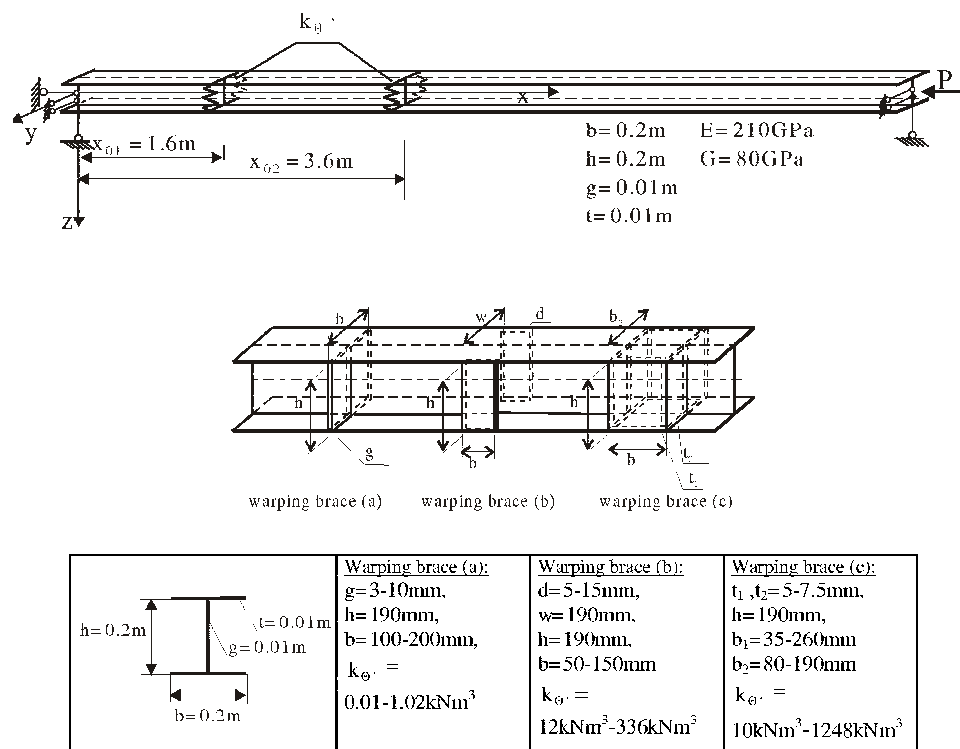


Fig. 8. Axially compressed thin-walled I-section column with two warping restraints.  
 Rys. 8. Pręt dwuteowy obciążony osiowo z dwoma usztywnieniami ograniczającymi spaznienie

The influence lines of the variation of torsional buckling load due to the location of a new stiffener with the unit warping stiffness are drawn (Fig. 9). The lines are related to the critical buckling load of each of analysed columns. These lines show that the points on the column where the warping stiffeners are most effective in increasing the buckling load are near supports.

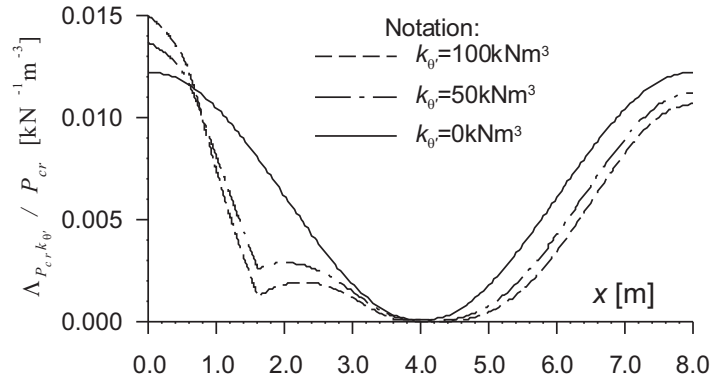


Fig. 9. The influence line of the relative variation of the torsional critical load due to the location of the restraint with unit warping stiffness for the column without any restraints and with two restraints of stiffness  $k_{\theta'} = 50 \text{ kNm}^3$ , and  $k_{\theta'} = 100 \text{ kNm}^3$  situated at  $x = 1.6 \text{ m}$  and  $x = 3.6 \text{ m}$

Rys. 9. Linia wpływu względnej wariacji obciążenia krytycznego wybożenia skrętnego w zależności od dodania usztywnienia ograniczającego spaczenie o jednostkowej sztywności dla słupa bez usztywnień i z dwoma usztywnieniami o sztywności  $k_{\theta'} = 50 \text{ kNm}^3$  i  $k_{\theta'} = 100 \text{ kNm}^3$  usytuowanymi w przekrojach  $x = 1.6 \text{ m}$  i  $x = 3.6 \text{ m}$

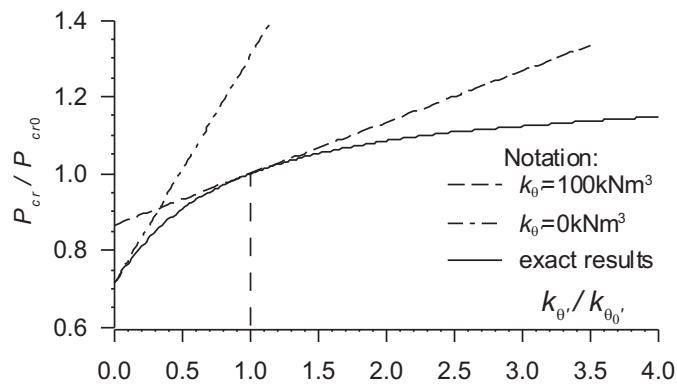


Fig. 10. The linear approximation of the exact relation of the relative torsional critical load due to a change of the stiffness of warping restraint at  $x = 1.6 \text{ m}$  and  $x = 3.6 \text{ m}$ , for a column without any restraints, and with restraints  $k_{\theta'} = 100 \text{ kNm}^3$ .

Rys. 10. Aproksymacja liniowa ścisłej zależności względnego obciążenia krytycznego wybożenia skrętnego w zależności od zmiany sztywności usztywnień ograniczających spaczenie o współrzędnych  $x = 1.6 \text{ m}$  i  $x = 3.6 \text{ m}$ , dla słupa bez usztywnień i z usztywnieniami o sztywności  $k_{\theta'} = 100 \text{ kNm}^3$

Next, the linear approximation of the exact relation of the critical load due to the variation of stiffeners stiffness is examined. The approximation is determined according Eq.(4.1) for both the column without stiffeners and when the column has two warping stiffeners of the stiffness equal to  $k_{\Theta'} = 100 \text{ kNm}^3$  (Fig. 10). The approximation is related to the critical load for the column with two stiffeners of the stiffness  $k_{\Theta'} = 100 \text{ kNm}^3$ .

Next, the critical load change due to the change of location of the stiffener at  $x=1.6 \text{ m}$  is investigated. A linear approximation of the relation between relative buckling load and the brace position is found using Eq. (4.2). A comparison of the exact and the approximated results is shown in Fig. 11.

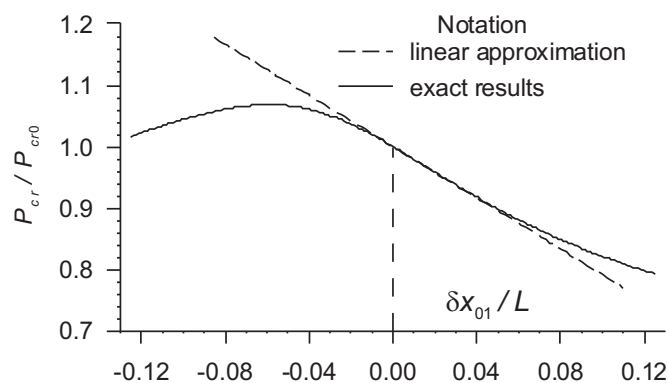


Fig. 11. The linear approximation of the exact relation of the relative torsional critical load of the column with two restraints due to a change of location of warping restraint  $k_{\Theta'} = 100 \text{ kNm}^3$  at the position  $x = 1.6 \text{ m}$ .

Rys. 11. Aproksymacja liniowa ścisłej zależności względnego obciążenia krytycznego wyoboczenia skrętnego słupa z dwoma usztywnieniami ograniczającymi spaczenie w zależności od zmiany położenia usztywnienia o współrzędnej  $x = 1.6 \text{ m}$  i o sztywności  $k_{\Theta'} = 100 \text{ kNm}^3$

### 5.3. COLUMN WITH TORSIONAL STIFFENERS

A simply supported I-section column with two torsional stiffeners positioned in the cross-sections at  $x = 1.6 \text{ m}$  and  $x = 3.6 \text{ m}$  is investigated (Fig. 12). It is assumed that the stiffness of the stiffeners is  $k_{\Theta_0} = 10 \text{ kNm/rad}$  and  $100 \text{ kNm/rad}$ . The critical load for the column without any stiffeners is  $2047.5 \text{ kN}$  and for the column with stiffeners of stiffness  $10 \text{ kNm/rad}$ , this load is  $P_{cr_0} = 3207.5 \text{ kN}$  and for stiffness  $100 \text{ kNm/rad}$ , this load is  $3717.3 \text{ kN}$ .

Deformation of column is related with deformation of lateral braces as purlins or wall rails resting on the column. On the assumption that connectors between lateral brace and column are stiff enough and are able to carry arising forces the rotation of lateral brace is interrelated with torsion of the column. Then the magnitude of stiffness



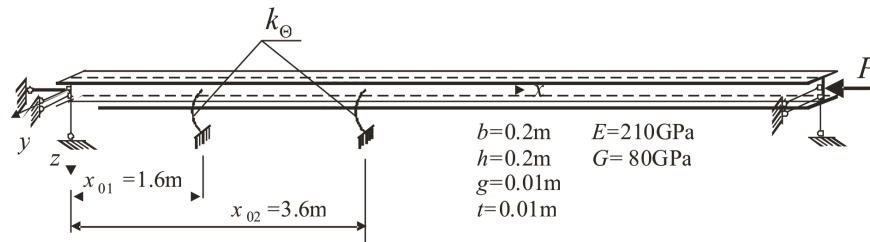


Fig. 12. Axially compressed thin-walled I-section column with two torsional restraint's.  
Rys. 12. Pręt dwuteowy obciążony osiowo z dwoma usztywnieniami ograniczającymi skręcenie

of torsional brace could be estimated as at least  $2EI_{br}/L_{br}$  for symmetrical deformation of one bay lateral element or as  $4EI_{br}/L_{br}$  for restraining elements fixed at one end or in the case of middle support of two span lateral brace  $6EI_{br}/L_{br}$ :

$$(5.2) \quad k_{\Theta} = \frac{M_{0br}}{\Theta} = \frac{2EI_{br}}{L_{br}}, \quad k_{\Theta} = \frac{4EI_{br}}{L_{br}}, \quad k_{\Theta} = \frac{6EI_{br}}{L_{br}}.$$

were  $(\dots)_{br}$  denotes values for brace. Then the magnitude of torsional brace stiffness is at least 5-30kNm/deg (280-1500kNm/rad) for 3-6 m long lateral brace.

The influence line of the variation of the torsional critical load due to the location of additional torsional stiffener of unit stiffness is shown in Fig. 13. All influence lines are related to the critical torsional load of each column. The lines show that in the middle of unbraced part of the column, the torsional stiffeners are most effective in increasing the buckling load.

Let us assume that the stiffness of torsional stiffeners increases. By means of the sensitivity analysis the linear approximation of the exact relation of the critical load due to some variations of stiffeners stiffness may be found (Eq.4.1). The approximation is determined for the column with stiffeners of stiffness  $k_{\theta 0} = 10$  kNm/rad (Fig. 14). The approximation is related to critical load of torsional buckling for the column. The critical load of torsional buckling variation due to the change of location of the stiffener placed at  $x = 3.6$  m was also investigated. The function of linear approximation is evaluated by means of Eq.(4.2). It was assumed that the stiffness of both stiffeners is 20 kNm/rad ( $P_{cr0} = 3381.4$  kNm). A comparison of the exact and the approximated results is shown in Fig. 15.

## 6. CONCLUSIONS

The sensitivity analysis makes it possible to find the graphs describing the influence of the location of the bracing elements along the column on the critical load. In the presented examples, the influence line of the torsional buckling load due to the location of the unit warping stiffeners allows to conclude that the location of stiffeners close to

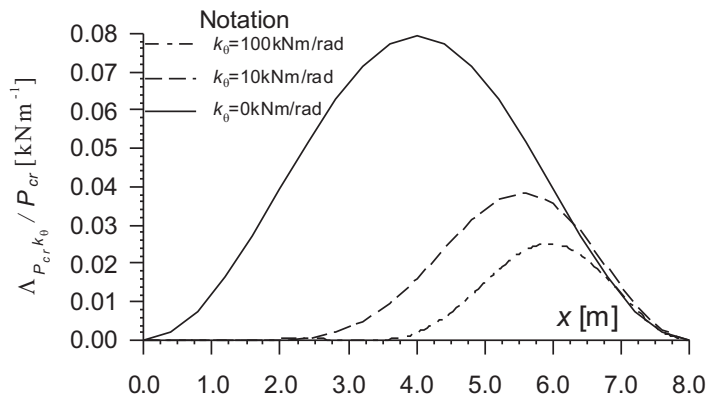


Fig. 13. The influence line of the relative variation of the torsional critical load due to the location of torsional restraints with unit stiffness for the column without any restraints, and with two restraints of the stiffness  $k_{\theta} = 10 \text{ kNm/rad}$ , and  $k_{\theta} = 100 \text{ kNm/rad}$ .

Rys. 13. Linia wpływu względnej wariacji obciążenia krytycznego wyboczenia skrętnego w zależności od dodania następnego usztywnienia ograniczającego skręcenie o jednostkowej sztywności dla słupa bez usztywnień i z dwoma usztywnieniami o sztywności  $k_{\theta} = 10 \text{ kNm/rad}$  i  $k_{\theta} = 100 \text{ kNm/rad}$

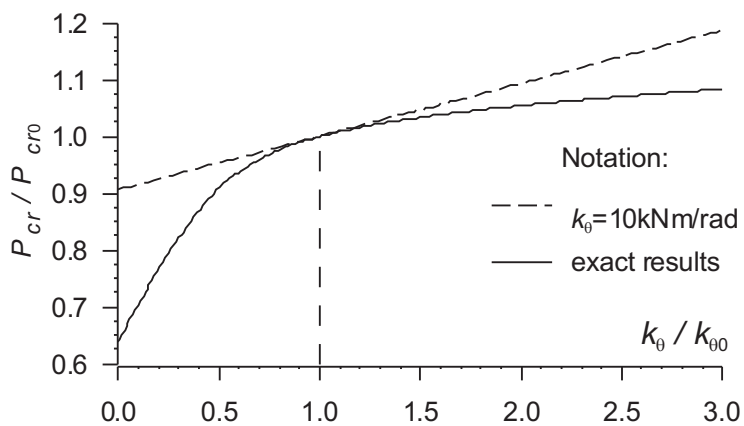


Fig. 14. The linear approximation of the exact relation of the relative torsional critical load due to a change of the stiffness of torsional restraint at  $x = 1.6 \text{ m}$  and  $x = 3.6 \text{ m}$  of the stiffness  $k_{\theta} = 10 \text{ kNm/rad}$ .

Rys. 14. Aproksymacja liniowa ścisłej zależności względnego obciążenia krytycznego wyboczenia skrętnego w zależności od zmiany sztywności usztywnień ograniczających skręcenie o współrzędnych  $x = 1.6 \text{ m}$  i  $x = 3.6 \text{ m}$  i o sztywności  $k_{\theta} = 10 \text{ kNm/rad}$

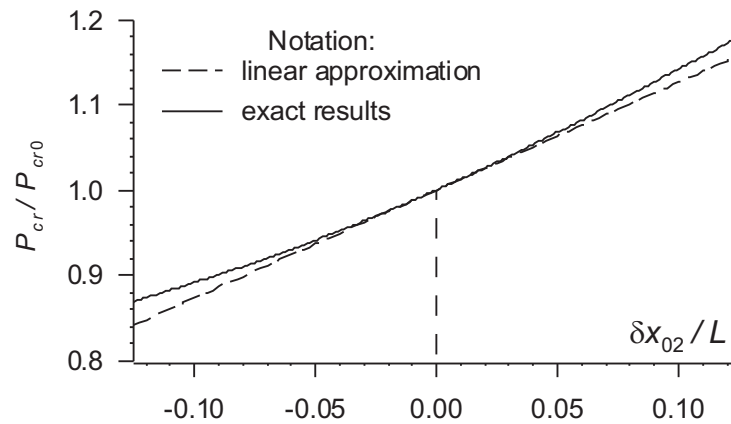


Fig. 15. The linear approximation of the exact relation of the relative torsional critical load of the column with two torsional restraints due to a change of the location of torsional restraint at  $x=3.6$  m of the stiffness  $k_{\theta} = 20$  kNm/rad.

Rys. 15. Aproksymacja liniowa ścisłej zależności względnego obciążenia krytycznego wybożenia skrętnego słupa z dwoma usztywnieniami ograniczającymi skręcenie w zależności od zmiany położenia usztywnienia o współrzędnej  $x=3.6$  m i o sztywności  $k_{\theta} = 20$  kNm/rad

the supports is of primary significance in increasing the torsional critical load. For a column with torsional stiffeners, the influence line of torsional buckling load makes it possible to conclude that the location of stiffeners near the middle of the unrestrained part of the column causes the most effective increase of the critical load. In the case of column with lateral bracing, one can conclude that the location of braces near the middle of the unrestrained part of the column causes the most effective increase of the flexural critical load. A comparison of the relative influence lines of the buckling load with respect to warping stiffeners allows to conclude that the lines are of equal significance for columns with or without stiffeners. In the case of torsional stiffeners and lateral bracing, the relative influence lines decreases with an increase of the stiffeners stiffness. The sensitivity analysis method may be helpful in the column design allowing to place the stiffeners most effectively. The sensitivity analysis makes it possible to find the approximation of the exact relation between the buckling load and the stiffness and location of the brace elements.

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## ANALIZA WRAŹLIWOŚCI SIŁY KRYTYCZNEJ SŁUPA CIENKOŚCIENNEGO ZE STĘŻENIAMI

### Streszczenie

Wyznaczono pierwszą wariację siły krytycznej wyboczenia słupa o przekroju bisymetrycznym otwartym przy wariacji sztywności i położenia stężeń. Przyjęto założenia klasycznej teorii prętów cienkościennych o nieodkształcalnym przekroju poprzecznym. Analizowano zarówno stężenia boczne jak i stężenia ograniczające spaczenie oraz skręcenie przekroju poprzecznego. Stężenia modelowano jako podpory sprężyste. W przykładzie numerycznym zbadano wrażliwość siły wyboczenia skrętnego i giętego słupa o przekroju dwuteowym przy wariacji sztywności i położenia stężeń. Wyznaczono funkcje podcałkowe wrażliwości opisujące wpływ wprowadzenia wzdłuż długości pręta dodatkowych stężeń o jednostkowej sztywności na wartość siły krytycznej. Ponadto wyznaczono liniową aproksymację względnej wartości sił krytycznych pręta przy wariacji sztywności i położenia stężeń.

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