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FINITE ELEMENT ANALYSIS OF PLATE ON LAYERED TENSIONLESS FOUNDATION

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The method of calculations of a thick plate on the two-parameter layered foundation by the finite element method is presented. The numerical model allows to add a few (number of) foundation layers. The expressions for the element stiffness matrices of the foundation are based on 18-node zero-thickness interface elements. For modelling of thick plates the 9-node Mindlin element of the Lagrange family is used. The formulation of the problem takes into account the shear deformation of the plate and unilateral contact conditions between plate and foundation. The tensionless character of the foundation is achieved by removing from the global stiffness matrix the appropriate part of foundation stiffness attached to the node being in the separation stage. The advantages of the proposed algorithm are illustrated by numerical examples.

Key words: Mindlin plate element, layered two-parameter foundation, unilateral contitions.

1. Introduction

In analysis of plates resting on the elastic foundation using the Winkler model, a single parameter k_0 is used to describe beahaviour of the foundation. In that method, it is assumed that deflection at each point is proportional to the pressure applied at the point and completly independent from the pressures or deflections occuring at the neighbouring points along the foundation. A model of such medium can be represented by an infinite row of closely spaced elastic springs, each of them being deformed only by the pressure acting on it, without any shear deformations between them.

Filonenko-Borodich [1] and Pasternak [2] attempted to make the classical Winkler more realistic model postulating a two-parameter model. Their model takes into account the effect of shear interaction among adjacent points in the foundation. In this model the shear parameter has to be determined experimentally. Vlasov and Leont'ev [3] have introduced another arbitrary parameter, γ, dependent on soil material and thickness of the soil layer and suggested an approximate value of γ between 1 and 2. However, they did not report the method of determining this parameter. In the work of Vallabhan and Daloglu [4], it has been shown how the soil parameter, γ, can be estimated using an

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iterative computational procedure for plates. The elastic foundation modulus E_s was there a linear function throught the thickness of the foundation z

$$E_s(z) = E_1 + (E_2 - E_1)\frac{z}{H} \tag{1.1}$$

where $E_1 = E_s(z = 0)$ and $E_2 = E_s(z = H)$ are the elasticity modulus at the top and bottom of the foundation, respectively. In the paper of Çelik and Omurtag [5] a quadratic variation of elasticity modulus E_s was discussed, given as

$$E_s(z) = E_1 + (E_2 - E_1)\frac{z^2}{H^2} \tag{1.2}$$

Çelik and Saygun [6], Ozgan and Daloglu, Ozgan and Omurtag [7, 8], Buczkowski and Torbacki [9] have presented finite element technique where the material properties of the soil are taken into account to incorporate the surrounding effect. In the last paper the response of the foundation was modelled by using an 18-node quadratic interface elements of zero-thickness. The authors concluded that the lagrangian quadratic 9-node elements can be an optimal choice for modelling thick plates interacting with non-homogeneous foundation. The elastic plates of rectangular shape resting on a tensionless Winkler foundation was analyzed by Celep [10]. The static and dynamic analysis of a circular plate on a two-parameter tensonless foundation was recently investigated using Galerkin's approximation technique by Celep and Güler [11]. The problem of tensionless elastic foundation under flexible rectangular Mindlin plates was also studied by Mishra and Chakrabarti [12]. The tensionless character of the foundation was achieved by employing a set of uncorrelated elastic springs attached to each node of the plate and by making these springs active in compression stage only. A unilateral plate contact with the elasto-plastic one-parameter Winkler foundation model was presented by Lewandowski and Świtka [13]. In the paper of Eratll and Aköz [14], a solution for Reissner plates on Winkler foundation based on the Gâteaux derivative theory was formulated. Özçelikörs *et al.* [15] also used the Gâteaux differential method combined with the classical Hellinger-Reissner and Hu-Washizu variational formulations in order to obtain a solution for interaction of orthotropic Kirchhoff plate and orthotropic Pasternak foundation. The mixed Galerkin-perturbation technique to nonlinear bending analysis of rectangular Reissner-Mindlin plate with free edges, resting on Pasternak elastic foundation, was recently attempted by Shen [16]. In the work of Feng and Owen [17] a coupled finite element and boundary element procedure to analyse a plate-foundation problem was described in which the boundary element equations of the foundation were not explicitly assembled with the finite element equations, but an iterative procedure was proposed to obtain the final coupled solution instead. The discrete singular convolution method was developed by Civalek and Acar [18] for static analysis of thick plates bending on the Pasternak foundation for different boundary conditions and arbitrary edges.

Analysis of interaction of thick plate with arbitrary layered foundation is a very complex engineering problem. The foundation consist of several layers of varying thickness and the kinematic constraints for each of the layers must be ensured. One of the first satisfactory numerical solution of rectangular plates resting on layered foundatiohn was obtained by Fraser and Wardle [19]. In their paper solutions were presented for the displacements and maximum bending moments for uniformly loaded raft of arbitarary rigidity resting on homogeneous isotropic multi-layered medium. Kolar and Nemec [20] developed a 3D-FEM model allowing for the contribution from the soil medium located outside the area of the plate. In their model, the effect of the reaction of the foundation outside the plate was involved by the use of the elastic springs distributed along the plate. In the model, the deflection outside the contact area between plate and foundation was approximated by the assumed exponential function. Sadecka [21] introduced modifications in the Kolar-Nemec model. In her model infinite elements for modelling the behaviour of layered foundation outside the plate were used. Wang *et al.* [22] presented a semi-analytical method combined with finite element for analyzing plates on layered soils by applying Hankel transforms. Almeida and Paiva [23] recently proposed the boundary element method to analyze of static soil-pile interaction in layered soil. In their study, the authors Kim, Lim and Cho [24] presented a solution for three-dimensional elastodynamics interaction of surface fundation on a layered half-space in which they adopted a coupling method that combines the finite element for surface foundation and the bounadry element for the layered half-space. In Ai nad Yue [25] the analytical method using the Hankel transforms, combined with the finite difference method, has been recently adopted for analyzing the axially loaded single piles in multilayered non-homogeneous medium subjected to a vertical point load. Andersen and Clausen [26] discussed a BEM solution for dynamic soil-structure interaction (a rigid foundation on a layered subsoil) whereas the solution based on domain-transformation method and the Green's function for layered viscoelastic half-space.

The main aim of this paper is to present a new method for the modelling of multi-layered foundation using zero-thickness interface elements. In Section 2 we begin with short description of the two-parameter foundation model and the finite element procedure in order to determine the stiffness matrices of the finite interface-foundation element. In Section 3 a numerical example is presented in order to illustrate the validity of the method proposed. In the final section some conclusions are drawn.

2. FINITE ELEMENT PROCEDURE

In this section we present short expressions for the element stiffness matrix and the soil reactions based on quadratic Lagrange 9-node Mindlin plate (nodal dof: w – transverse displacement in the direction z and θ_x, θ_y – the normal rotations about the x- and y-axis respectively) and 18-node zero-thickness foundation elements [27]. We

recommend here the nine-node Mindlin plate element of the Lagrange family in which 3×3 Gauss integration scheme is employed for both the bending, and shear deformation to handle extremely thin plates for very large length-thickness ratios up to $b/t = 10^6$ [9]. (An alternative proposal is the use of the heterosis element which represents a synthesis of the serendipity type of the interpolation which is used on displacement only, and a lagrangian one on rotations, see Reference [28].)

The present formulation is based on the theory of the plates which enables the transverse shear deformations. In this theory the displacements w and both rotations β_x and β_y at the plate midsurface are assumed independent variables. Moreover, the concept of isoparametric C^0 finite plate element can be directly constructed. According to the small bending theory, the displacement components of a point of coordinates x, y and z are

$$u = -z\beta_x(x, y), \quad v = -z\beta_y(x, y), \quad w = w(x, y) \tag{2.1}$$

where w is the transverse displacement and $\beta_x(x, y)$ and $\beta_y(x, y)$ are the rotations of the normal to the undeformed middle surface. In the Kirchhoff plate theory shear deformations are excluded, then $\beta_x = w_{,x}$ and $\beta_y = w_{,y}$, see for details Reference [29].

2.1. Formulation of the foundation

A two-parameter model for a plate resting on elastic foundation including a shear effect of the foundation is investigated. In this model the expression relating the pressure and corresponding deflection of the foundation is [2]

$$p = k_0 w - k_1 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \tag{2.2}$$

where p, w and k_1 are the surface pressure, the transverse deflection, and the shear modulus of the foundation (second foundation parameter) respectively; k_0 is modulus of subgrade reaction of the soil (first or Winkler foundation parameter).

The strain energy expressions for the Winkler foundation and the energy associated with the second foundation parameter k_1 are

$$U_F = \frac{1}{2} \int_A [w(x, y)]^T k_0 [w(x, y)] \, \mathrm{d}x\mathrm{d}y + \frac{1}{2} \int_A \gamma_{\mathrm{F}}^{\mathrm{T}} k_1 \gamma_{\mathrm{F}} \mathrm{d}x\mathrm{d}y \tag{2.3}$$

The second term in the above equation can be regarded as the strain energy in shear associated with the parameter k_1, where the shear strain vector γ_{F} can be written as

$$\gamma_{\mathrm{F}} = \left\{ \begin{array}{c} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{array} \right\} \tag{2.4}$$

For the foundation elements, the soil tractions and relative soil displacements $w(x, y)$ are evaluated for nodes situated between the top and bottom planes of the interface element; for details we refer to Reference [27]. We express these relative transverse displacements as

$$w(x, y) = w(x, y)_{\mathrm{TOP}} - w(x, y)_{\mathrm{BOT}} = \mathbf{N}\mathbf{w} \tag{2.5}$$

where $\mathbf{w}$ is a vector of the nodal transverse displacements of the interface element defined as

$$\mathbf{w} = [w_1 \; w_2 \; ... \; w_9 \; w_{10} \; w_{11} \; ... \; w_{18}]^{\mathrm{T}} \tag{2.6}$$

and the matrix $\mathbf{N}$ is given by

$$\mathbf{N} = [N_1 \; N_2 \; N_3 \; ... \; N_9 \; -N_1 \; -N_2 \; -N_3 \; ... \; -N_9] \tag{2.7}$$

Where N_i is a shape function for node i. The matrix $\mathbf{N}$ includes non-differentiated shape functions of the thick nine-node plate Mindlin element is given, for example, in [29].

By substituting equation (2.5) into the strain energy expression (equation (2.3)) we obtain the total stiffness matrix of the foundation

$$\mathbf{K}_{ij} = k_0 \iint\limits_A N_j N_i \, dxdy + k_1 \iint\limits_A \left(\frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x} + \frac{\partial N_j}{\partial y}\frac{\partial N_i}{\partial y} \right) \mathrm{d}x\mathrm{d}y \tag{2.8}$$

Following the isoparametric concept, the stiffness submatrices linking nodes i and j of the interface element related to the first k_0 and the second foundation parameter k_1, $(\mathbf{K}_0)_{ij}$ and $(\mathrm{K}_1)_{ij}$ have the form

$$(\mathrm{K}_0)_{ij} = k_0 \int\limits_{-1}^{+1} \int\limits_{-1}^{+1} N_j N_i \det \mathbf{j} \mathrm{d}\xi \mathrm{d}\eta \tag{2.9}$$

and

$$(\mathrm{K}_1)_{ij} = \int_{-1}^{+1}\int_{-1}^{+1} \frac{k_1}{\det \mathbf{j}} \left(\frac{\partial N_j}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial N_j}{\partial \eta}\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial N_i}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial N_i}{\partial \eta}\frac{\partial y}{\partial \xi}\right)\mathrm{d}\xi\mathrm{d}\eta + \int_{-1}^{+1}\int_{-1}^{+1} \frac{k_1}{\det \mathbf{j}} \left(-\frac{\partial N_j}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial N_j}{\partial \eta}\frac{\partial x}{\partial \xi}\right)\left(-\frac{\partial N_i}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial \eta}\frac{\partial x}{\partial \xi}\right)\mathrm{d}\xi\mathrm{d}\eta \tag{2.10}$$

where the determinant of the transformation Jacobian matrix is expressed as

$$\det \mathbf{j} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \tag{2.11}$$

The reactions of foundation related to the first parameter k_0 and the second one k_1, that is the transverse force W and the shear forces (at the node i) $\mathbf{T} = (T_x, T_y)$ respectively, can be expressed as [9]

$$(W)_i = k_0 \int_{-1}^{+1}\int_{-1}^{+1} N_i w_i \det \mathbf{j}\mathrm{d}\xi\mathrm{d}\eta \tag{2.12}$$

and

$$(T_x)_i = k_1 \int_{-1}^{+1}\int_{-1}^{+1} \left(\frac{\partial N_i}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial N_i}{\partial \eta}\frac{\partial y}{\partial \xi}\right) w_i \mathrm{d}\xi\mathrm{d}\eta \tag{2.13}$$

$$\left(T_y\right)_i = k_1 \int_{-1}^{+1}\int_{-1}^{+1} \left(\frac{\partial N_i}{\partial \eta}\frac{\partial x}{\partial \xi} - \frac{\partial N_i}{\partial \xi}\frac{\partial x}{\partial \eta}\right) w_i \mathrm{d}\xi\mathrm{d}\eta \tag{2.14}$$

in which the expressions $\partial x/\partial \xi$, $\partial x/\partial \eta$, $\partial y/\partial \xi$ and $\partial y/\partial \eta$ are components of the Jacobian matrix.

The numerical integration of equations (2.9) and (2.10) for the quadrilateral element of rectangular shape with 3×3 sampling points leads to

$$(\mathrm{K}_0)_{ij} = \sum_{p=1}^{3}\sum_{q=1}^{3} N_i^T\left(\xi_p, \eta_q\right) k_0 N_j\left(\xi_p, \eta_q\right) \det \mathbf{j}\left(\xi_p, \eta_q\right) W_p W_q \tag{2.15}$$

and

$$(\mathrm{K}_1)_{ij} = \frac{b^2}{4}\sum_{p=1}^{3}\sum_{q=1}^{3}\frac{k_1}{\det \mathbf{j}\left(\xi_p,\eta_q\right)}\left(N_{i,\xi}\right)^2\left(\xi_p,\eta_q\right)W_pW_q + \frac{a^2}{4}\sum_{p=1}^{3}\sum_{q=1}^{3}\frac{k_1}{\det \mathbf{j}\left(\xi_p,\eta_q\right)}\left(N_{i,\eta}\right)^2\left(\xi_p,\eta_q\right)W_pW_q \tag{2.16}$$

with W_p and W_q being the integration weights.

The stiffness matrices of the interface element related to the first, k_0, and the second parameter, k_1, (equations (2.15 and (2.16)) were integrated, and their explicit forms are given in Reference [9].

2.2. Foundation parameters k_0 and k_1

The two parameters k_0 and k_1 in terms of the elastic contstants and the dimensions of the plate and the soil foundation were evaluated by Vallabhan, Straughan and Das [4]. These parameters applied to the foundation with a finite depth of soil, h, can be defined by

$$k_0 = \frac{E_0}{\left(1-\nu_0^2\right)}\int_0^h \Psi'(z)^2\mathrm{d}z \tag{2.17}$$

and

$$k_1 = \frac{E_0}{2\left(1+\nu_0\right)}\int_0^h \Psi^2(z)\mathrm{d}z \tag{2.18}$$

with the mode function $\Psi(z)$ which can be obtained by using variational principles and applying the proper boundary conditions, such as $\Psi(0) = 1$ and $\Psi(h) = 0$, so due to Vallabhan *et al.* [4] this function is given as

$$\Psi(z) = \frac{\sinh\gamma\frac{h-z}{h}}{\sinh\gamma} \tag{2.19}$$

The generalized modulus of elasticity, E_0, and the Poisson ratio, ν_0, are defined by

$$E_0 = \frac{E_F}{1-\nu_F^2}, \quad \nu_0 = \frac{\nu_F}{1-\nu_F} \tag{2.20}$$

where E_F and ν_F are the modulus of elasticity and Poisson's ratio of the foundation, respectively, and h is the foundation depth. If the elasticity modulus E_s is constant through the thickness of the foundation and using the mode function $\Psi(z)$ as given in equation (2.19), the foundation parameters k_0 (equation (2.17)) and k_1 (equation (2.18)) become

$$k_0 = \frac{E_F(1-\nu_F)}{8h(1+\nu_F)(1-2\nu_F)} \frac{2\gamma \sinh 2\gamma + 4\gamma^2}{\sinh^2\gamma} \tag{2.21}$$

and

$$k_1 = \frac{E_F h}{16\gamma^2(1+\nu_F)} \frac{2\gamma \sinh 2\gamma - 2\gamma^2}{\sinh^2\gamma} \tag{2.22}$$

However, these parameters also depend on a coefficient, γ, which represents the rate of decrease of the displacement and the normal stresses in the vertical direction in the soil foundation. According to Vallabhan, Straughan and Das [4] the parameter γ can be expressed as

$$\gamma^2 = h^2 \frac{(1-2\nu_s)}{2(1-\nu_s)} \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left\{ \left(\frac{\partial w(x,y)}{\partial x}\right)^2 + \left(\frac{\partial w(x,y)}{\partial x}\right)^2 \right\} \mathrm{d}x\mathrm{d}y}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} w^2(x,y)\,\mathrm{d}x\mathrm{d}y} \tag{2.23}$$

For plates on the elastic foundation, the dimensionless parameter, γ, can be evaluated using an iterative procedure after the determination of the vertical displacement $w(x, y)$. At the end of every iteration a criterion is monitored to see whether the iterative solution converged within the tolerance.

In order to analyze cases in which the modulii of elasticity of the foundation E_s (equation (1.1)) can vary linearly in the vertical direction from E_1 at the top to E_2 at the bottom, expressions for the foundation parameters k_0 and k_1 can be modified to the following form

$$k_0 = \frac{1-\nu_F}{8h(1+\nu_F)(1-2\nu_F)} \frac{E_1(2\gamma \sinh 2\gamma + 4\gamma^2) + (E_2 - E_1)(\cosh 2\gamma - 1 + 4\gamma^2)}{\sinh^2\gamma} \tag{2.24}$$

and

$$k_1 = \frac{h}{16\gamma^2(1+\nu_F)} \frac{E_1(2\gamma \sinh 2\gamma - 2\gamma^2) + (E_2 - E_1)(\cosh 2\gamma - 1 - 2\gamma^2)}{\sinh^2\gamma} \tag{2.25}$$

When the elasticity modulus E_s (equation (1.2)) changes quadratically through the depth of the foundation, the foundation parameters k_0 and k_1 can be evaluated by

(2.26)
$$k_0 = \frac{1-\nu_F}{(1+\nu_F)(1-2\nu_F)} \frac{3[E_2 + E_1(2\gamma^2-1)]\sinh 2\gamma + [E_2(4\gamma^2-3) + E_1(3+2\gamma^2)]2\gamma}{24\,h\,\gamma\sinh^2\gamma}$$

and

$$k_1 = \frac{h}{(1+\nu_F)} \frac{3[E_2 + E_1(2\gamma^2-1)]\sinh 2\gamma - [E_2(4\gamma^2+3) + E_1(2\gamma^2-3)]2\gamma}{48\,\gamma^3\sinh^2\gamma} \tag{2.27}$$

3. NUMERICAL APPLICATION

A computer program for the finite element analysis of bending Mindlin plates resting on a two-parameter elastic layered foundation is used to verify the accuracy of the proposed method. The finite element model allows to add a few foundation layers (Figure 1).

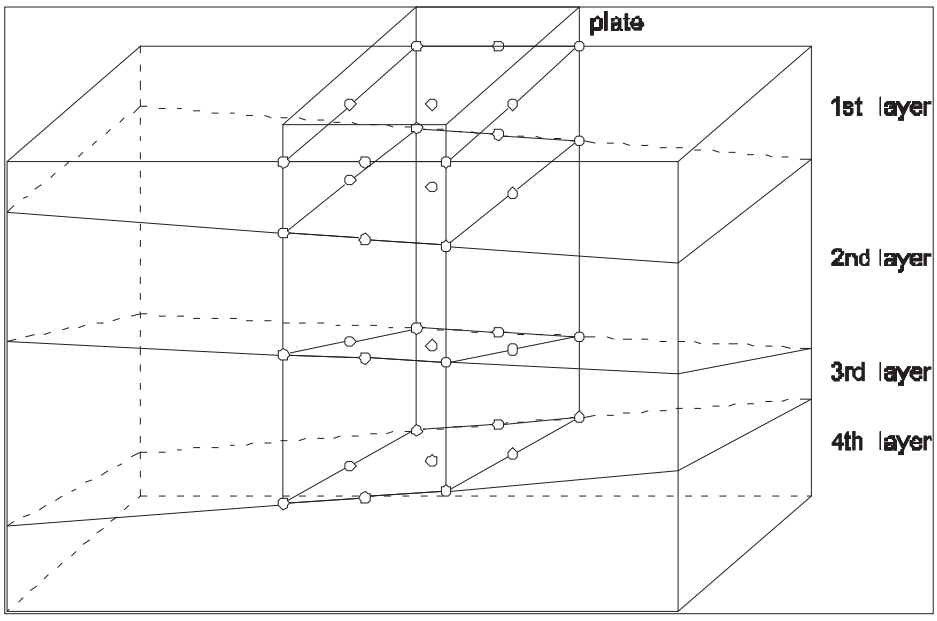

Fig. 1. The finite element model of the layered foundation.
Rys. 1. Model podłoża wielowarstwowego za pomocą elementów skończonych

3.1. Free rectangular plate on an unilateral homogeneous foundation

A free rectangular plate of the size of 9.6 × 7.2 m and thickness of $t = 0.2$ m resting on a two-parameter homogeneous soil medium subjected to uniformly distributed load $q = 1$ MPa on a central (painted in grey) square (Figure 2) is considered first.

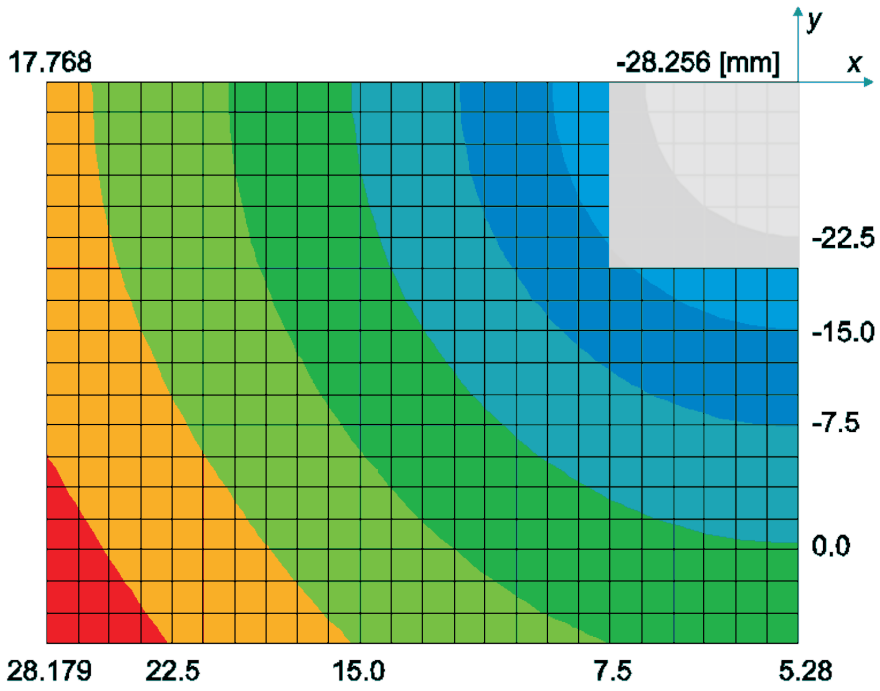

Fig. 2. Free rectangular plate on an unilateral Winkler foundation; (for $w > 0$ [mm] passive contact zone – separation or loss of contact).
Rys. 2. Pływająca prostokątna płyta na jednostronnym podłożu Winklera; ($w > 0$ [mm] – kontakt pasywny – separacja lub utarata kontaktu)

Due to the symmetry only the quarter of the plate is taken into account. The 18 × 24 mesh consists of 432 quadratic Lagrange 9-node Mindlin plate elements. The calculations are performed for the first (Winkler) k_0 and the second foundation parameter k_1. Both the foundation parameters are assumed to be dependent on material properties and the depth of the soil medium (see equations (2.21) and (2.22)) as well as on the dimensionless parameter γ given by equation (2.23). The calculations are performed for the foundation depth, $h = 5$ m, the Poisson ratios $\nu_s = 0.45$ and $\nu_p = 0.17$ for the soil (foundation) and plate, respectively. The Young modulus of the plate is assumed $E = 23$ GPa. In order to enable comparison of our results with the first parameter only, with those reported by Lewandowski and Świtka [13] and Jirousek *at al.* [30], we made calculations for $E_1 = E_2 = 42$ MPa and γ as given in Table 1.

The results are in close agreement with those by Jirousek *at al.* [30], solved by a hybrid-Trefftz approach and by assuming the Winkler conditions there.

Table 1

The settlement of the plate along x-axis for uniformly distributed load [mm], (k_0 [MPa/m], k_1 [MPa m], $h = 5$ [m], $E_1 = E_2 = 42$ [MPa]).
Obniżenie płyty wzdłuż osi w przypadku obciążenia stałym ciśnieniem [mm], (k_0 [MPa/m], k_1 [MPa m], $h = 5$ [m], $E_1 = E_2 = 42$ [MPa])

x [m]	Winkler model ($k_0 = 20$, $k_1 = 0$)	$\gamma = 1.1316$ ($k_0 = 32.7862$, $k_1 = 30.0426$)	$\gamma = 2.1327$ ($k_0 = 39.1322$, $k_1 = 16.4144$)
0.0	-28.256	-12.962	-5.234
0.6	-24.571	-10.978	-4.759
1.2	-19.961	-7.952	-3.269
1.8	-12.272	-4.283	-1.328
2.4	-4.674	-0.427	0.261
3.0	1.459	3.190	1.389
3.6	7.368	6.487	2.474
4.2	12.491	8.423	3.427
4.8	17.768	9.661	4.189

It can be noted from Table 1 that, for larger parameter k_0 , the settlement of foundation is smaller. The larger parameter γ the smaller the active contact zone ($w < 0$), see Figure 3. This is the result of resistance of the foundation due to influence of the second parameter. In particular, for very small values of the parameter k_1 the behaviour of the foundation approaches the Winkler foundation model for which the second parameter k_1 is assumed zero. For the Winkler approach, the value of maximum settlement at the center of the plate is equal to $w = 28.256$ [mm] (see another results from Table 1 and Figure 3 to compare).

3.2. FREE RECTANGULAR PLATE ON A LAYERED FOUNDATION

A free rectangular plate of the size 9.144×12.192 m and thickness of 0.1524 m, subjected to uniformly distributed load, were analyzed next (Figure 4). The size of the fundation and its depth are: 27.432×30.48 m and $h = 3.048$ m. It was assumed the Poisson ratio equals to $\nu_F = 0.25$ and $\nu_p = 0.20$ for the soil and plate, respectively. The same problem in which the elasticity modulus of the foundation E_2 at the bottom is equal to E_1 at the top was considered by Çelik and Saygun [6] and Vallabhan *at al.* [4].

The results are obtained for $E_1 = 68950$ kN/m^2 and $E_p = 20685$ MN/m^2 for the top of the foundation and the plate, respectively. The foundation parameters k_0 and k_1 are assumed dependent on material properties and the depth of the foundation, as well as on the dimensionless parameter γ given by equation (2.23). The vertical displacements at the centre of the plate are shown in Table 2.

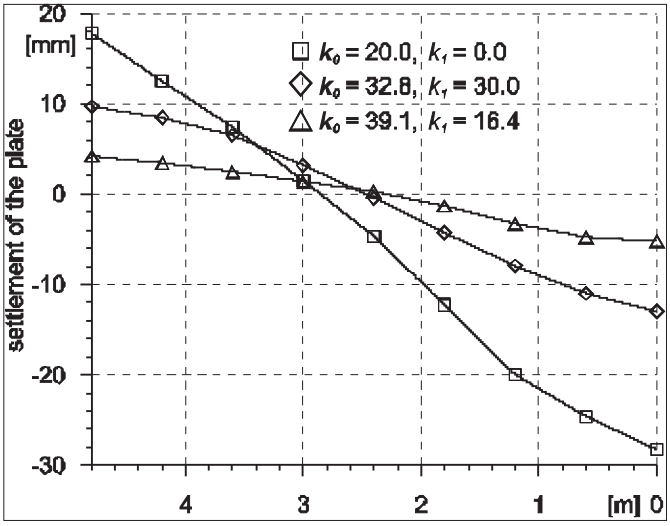

Fig. 3. The settlement of the plate along x-axis for uniformly distributed load.
Rys. 3. Obniżenie płyty wzdłuż osi x w przypadku obciążenia stałym ciśnieniem

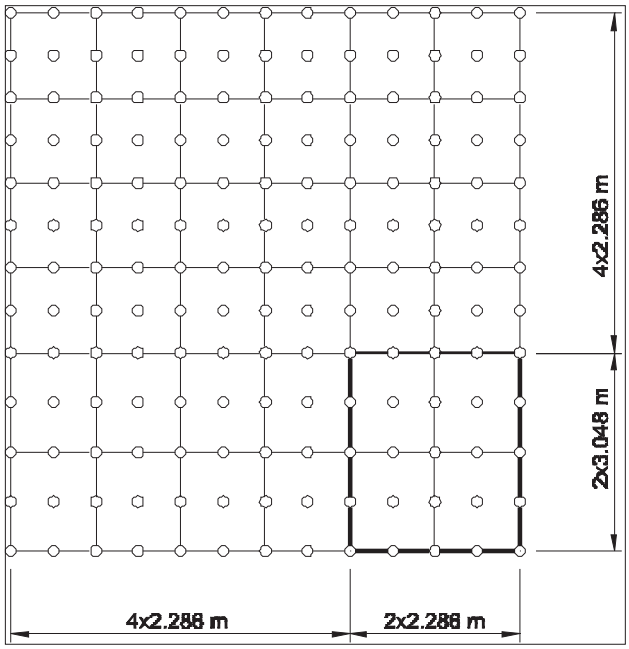

Fig. 4. Free rectangular plate on elastic foundation (only one quadrant is discretized).
Rys. 4. Pływająca prostokatna płyta na podłożu sprężystym (pokazano tylko jedną ćwiartkę)

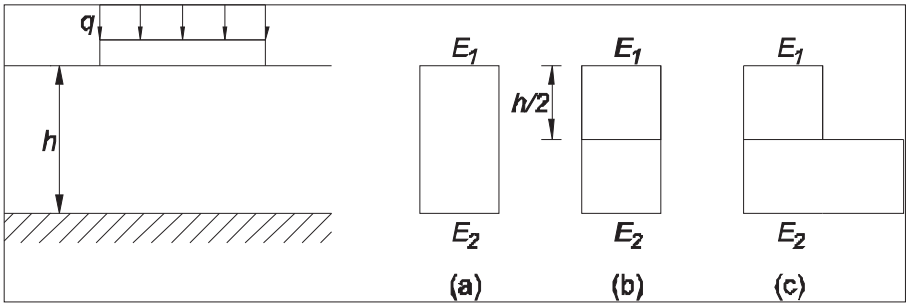

Fig. 5. Free plate: (a) one layer, $E_2 = E_1$, (b) two layers, $E_2 = E_1$, (c) two layers, $E_2 = 2E_1$.
Rys. 5. Płyta pływająca, $E_2 = E_1$, (b) dwie warstwy, $E_2 = E_1$, (c) dwie warstwy, $E_2 = 2E_1$

Table 2

The vertical displacement at the centre of the plate for uniformly distributed load.
Pionowe przemieszczenie centralnego punktu płyty w przypadku obciążenia stałym ciśnieniem

h (m)	Reference	k_0(kN/m^3)	k_1(kN/m)	γ	w (cm)
3.048	Present, Figure 5(a)	27212	81789	0.5844	0.0734
	Present, $E_2 = E_1$, Figure 5(b)	54423/54423 top/bottom	40895/40895 top/bottom	0.5844	0.0733
	Present, $E_2 = 2E_1$, Figure 5(c)	54433/108865 top/bottom	39825/79650 top/bottom	0.5947	0.0609

3.3. FREE SQUARE PLATE ON A LAYERED FOUNDATION (FIGURE 6)

A square thick plate of length $a = 5$ m and thickness of $t = 0.25$ m or $t = 1.25$ m resting on two-layered foundation of size 10×10 m is considered. The same example was also analyzed by Sadecka [21]. The numerical solution obtained by a weighting method for the isotropic semi-infinite medium was presented by Fraser and Wardle [19]. In the Kolar-Nemec's model, the deflection outside the contact area between plate and foundation is approximated by the assumed exponential function. Sadecka introduced modifications in the Kolar-Nemec's model. In her model infinite elements were used for modelling of the layered foundation outside the plate. A quarter of plate division into finite elements is shown in Figure 6. The whole plate is loaded by a uniformly distributed load $q = 1$ MPa.

The foundation is divided into two layers of thickness $h_2 = 3$ m and $h_1 = 7$ m for the top and the bottom, respectively (Figure 7).

The Poisson's ratio for both the layers is $\nu_F = 0.3$. The material properties of the plate are as follows: Young's modulus $E_p = 21000$ MPa, Poisson's ratio $\nu_p = 0.167$. The generalized modulus of elasticity from equation (2.20) $E_0 = E_p/500$ and $E_0 = E_p/1000$ are chosen for calculations. According to equations (2.21) and (2.22), the following foundation parameters are obtained for both the foundation layers: in the

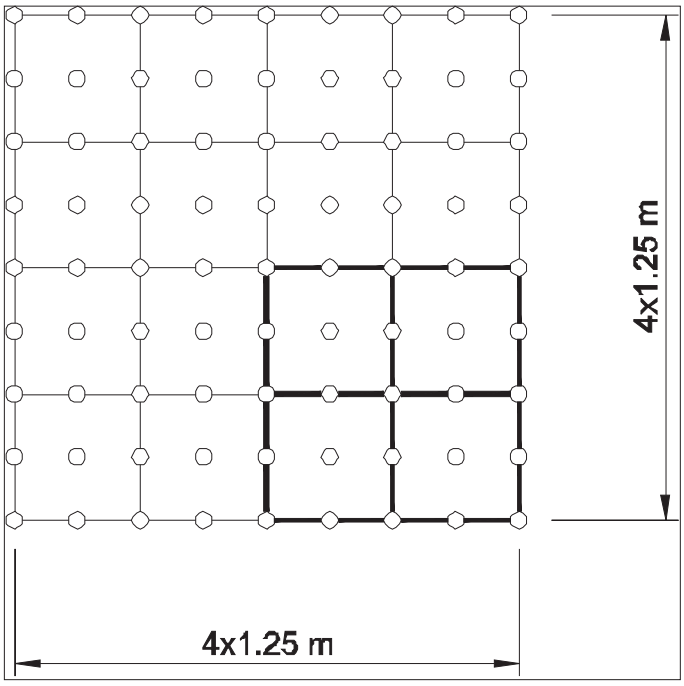

Fig. 6. Arrangement of square plate on elastic foundation.
Rys. 6. Aranżacja płyty kwadratowej na podłożu sprężystym

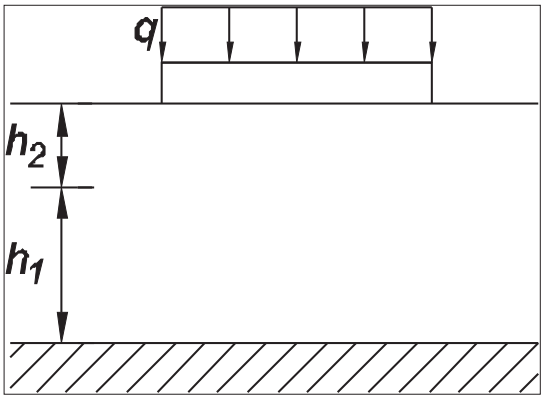

Fig. 7. Thick plate on two-layered foundation.
Rys. 7. Płyta gruba na podłożu dwuwarstwowym

case of $E_p/E_0 = 500$, $k_0 = 5.291$ MPa / m and $k_1 = 61.369$ MPa m, while in the case of $E_p/E_0 = 1000$, $k_0 = 2.646$ MPa / m and $k_1 = 30.684$ MPa m.

Selvadurai [31] suggests the following form for the modulus of foundation in the case of the Winkler model

$$k_0 = \frac{2E_F}{a \cdot \ln(1 + 2(H/a))} \tag{3.1}$$

where $H = h_1 + h_2$ is the thickness of the foundation and a is the length of the plate.

Table 3

The vertical displacement of the plate for uniformly distributed load (in [mm]).
Pionowe przemieszczenie płyty w przypadku obciążenia stałym ciśnieniem (w [mm])

E_p/E_0	(t/a)	γ	present Winkler	present two parameter	Sadecka [21]	Fraser and Wardle [19]
500	0.05	1.1249	10.1074	7.6014	7.8040	10.61
500	0.25	1.1249	9.8243	5.1059	5.3538	7.77
1000	0.05	1.1254	19.9589	14.4128	14.6173	21.21
1000	0.25	1.1254	19.6451	10.4276	10.5891	15.54

According to equation (30), for $E_p/E_0 = 500$, the Winkler parameter k_0 equals 9.4990 MPa/m, while in the case of $E_p/E_0 = 1000k_0$ it is equal to 4.7495 MPa/m for both the foundation layers. As expected, the maximum value of the settlement at the center of the plate for the Winkler approach is always greater than for the two-parameter one. Table 3 summarizes the results for both, the one- and two-parameter foundation models. The results obtained for the two-parameter model are closer to the solution of Sadecka [21] than the results obtained for semi-infinite medium by Fraser and Wardle [19].

3.4. FREE SQUARE PLATE ON A FOUR-LAYER FOUNDATION (FIGURE 8)

A thick plate of size 10×10 m and thickness of $t = 0.5$ m resting on a four-layer foundation of size 20×20 m is considered. The example was adopted from Fraser and Wardle [19] and Wang *et al.* [22] where results for elastic halfspace were given. The material constants of the plate are as follows: Young's modulus $E_p = 15000$ MPa, Poisson's ratio $\nu_p = 0.2$. Due to the symmetry, only a quarter of the square plate is taken into consideration. The whole plate was loaded by a uniformly distributed load $q = 0.1$ MPa.

The foundation is divided into four layers of thickness 10 m (see Figure 9). The Poissons' ratio for all layers is equal to $\nu_F = 0.3$. The Youngs' modulus, the foundation parameters k_0 and k_1 are given in Figure 9. The parameter γ is obtained to be equal to 1.2481 for each layer.

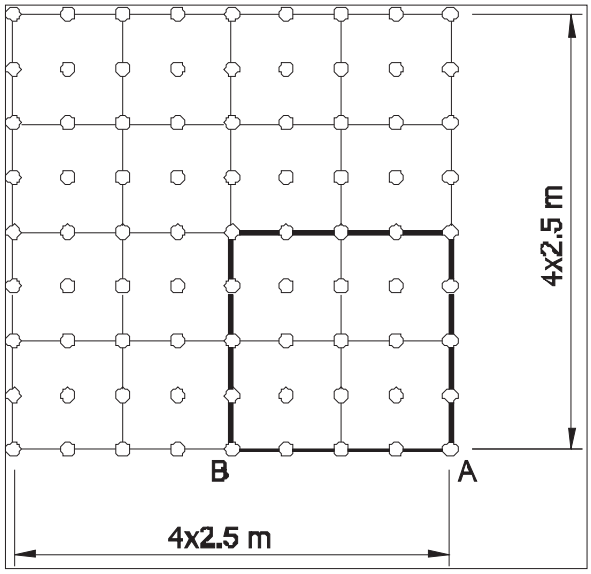

Fig. 8. Finite element discretization of square plate.
Rys. 8. Siatka elementów skończonych płyty kwadratowej

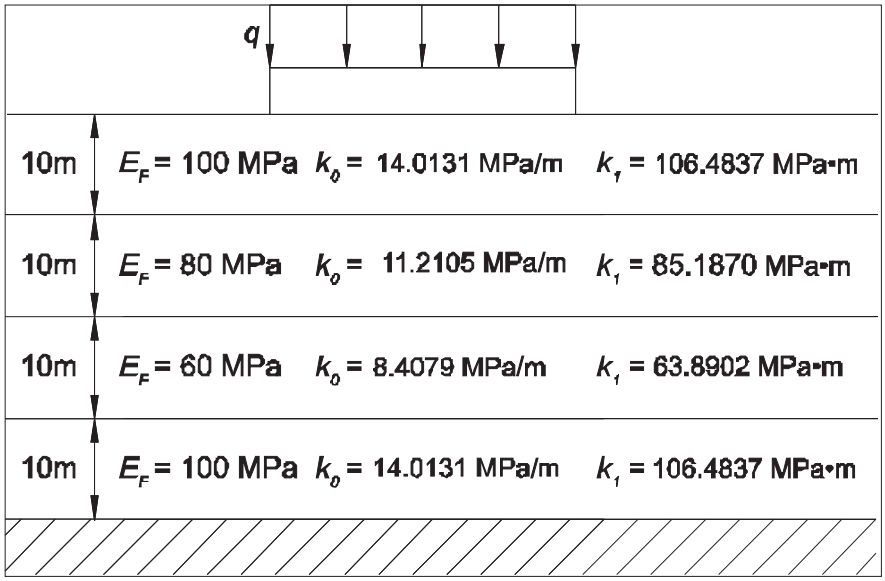

Fig. 9. Thick plate on four-layered foundation.
Rys. 9. Płyta gruba na podłożu czterowartwowym

Table 4

The vertical displacement of the plate for uniformly distributed load (in [mm]).
Pionowe przemieszczenie płyty w przypadku obciążenia stałym ciśnieniem (w [mm])

Point of plate	Present	Wang *at al.*[22] numerical	Fraser and Wardle [19] numerical	Fraser and Wardle [19] equivalent
A	0.0102	0.0120	0.0114	0.0107
B	0.0071	0.0089	0.0087	0.0078

The results are compared with those reported by Wang *at al.* [22], and the numerical results presented by Fraser and Wardle [19] (Table 4). The result are in good agreement with both the numerical results of the other authors.

4. CONCLUSIONS

The problem of bending plate resting on the two-parameter layered foundation was solved by using the interface elements of zero-thickness. The adopted model can be employed to analyze thick plates resting on foundation with any type of common boundary conditions and loading combinations. The method proposed may also take account of the surrounding effect outside the plate, as well as the unilateral contact conditions between the plate and foundation. A good agreement with the other results reported in the literature is worth noting.

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APPENDIX: TENSIONLESS FOUNDATION: ITERATIVE SOLUTION PROCEDURE

The solution of the problem of tensionless foundation is obtained by the incremental iterative Newton-Raphson procedure which goes as follows.

(i) *contact conditions*

$$w(x,y) = \left((\mathbf{w}^B + \Delta\mathbf{w}^B) - (\mathbf{w}^A - \Delta\mathbf{w}^A)\right)\mathbf{n} - g_0 \geq 0 \Rightarrow W = k_0 \iint_A [w(x,y)]\ \mathrm{d}x\mathrm{d}y \leq 0 \tag{1}$$

(a*ctive node*),

$$w(x,y) = \left((\mathbf{w}^B + \Delta\mathbf{w}^B) - (\mathbf{w}^A - \Delta\mathbf{w}^A)\right)\mathbf{n} - g_0 < 0 \Rightarrow W = 0\,(\text{p}\textit{assive node})$$

where

$w(x,y)$ – the actual value of the gap between the plate and the foundation,

$\mathbf{n}$ – the unit normal vector to be assumed constant,

W – the actual value of the normal contact force,

$\Delta\mathbf{w}$ – the value of an increment displacement due to load increment,

g_0 – the initial gap between the plate and the foundation.

The indices A and B correspond to the top and the bottom nodes of the zero-thickness interface element, respectively. In equation of constraint optimization (1) can be concisely written as the Kuhn-Tucker condition

$$W\, w(x,y) = 0, \quad W \leq 0, \quad w(x,y) \geq 0 \tag{2}$$

(ii) *for the* $(m+1)$ *external load increment* $\Delta\mathbf{R}_{m+1}$.

At the beginning of each load increment $(m+1)$, i.e. $\Delta\mathbf{R}_{m+1}$, the iteration scheme is based on the following parameters: the nodal displacements $\Delta\mathbf{w}_m$, the unbalanced forces $\boldsymbol{\Psi}^n_{m+1}$ and the nodal vector of internal forces $\Delta\mathbf{f}_{m+1}$. Thus within an increment of load $(m+1)$ we have

$$\Delta\boldsymbol{\Psi}^n_{m+1} = \Delta\mathbf{R}_{m+1} - \left[\Delta\mathbf{f}^n_{m+1} + \Delta\mathbf{W}^n_{m+1}\right], \tag{3}$$

where m denotes the number of load increment, n is the number of iteration at every increment, $\Delta\mathbf{f}_{m+1}$ is the nodal increment vector of internal forces, i.e., the resultant forces acting on each node from the surrounding elements which is calculated from the actual element stresses

$$\Delta\mathbf{f}_{m+1^n} = \int \boldsymbol{B}^T \Delta\boldsymbol{\sigma}^{\boldsymbol{n}}_{\boldsymbol{m+1}} dV, \tag{4}$$

while

$$\Delta\mathbf{W}^n_{m+1} = k_0 \iint_A \Delta\mathbf{w}^n_{m+1}\ \mathrm{d}x\mathrm{d}y. \tag{5}$$

is the increment vector of the normal contact forces.

To define $\mathbf{w}^n_{m+1}$ in expression (3) we can use the Taylor series expansion up to first-order and write

$$\Delta\boldsymbol{\Psi}^{n+1} = \Delta\boldsymbol{\Psi}^n + \frac{\partial(\Delta\boldsymbol{\Psi}^n)}{\partial(\mathbf{w}^n)}\Delta\mathbf{w}^n \cong \mathbf{0}. \tag{6}$$

For clarity we omit the superindex $(m+1)$ in the above expression. Substituting equations (3) and (4) to (6) lead to

$$\Delta\boldsymbol{\Psi}^{n+1}_{m+1} = \left[\Delta\mathbf{R}_{m+1} - (\Delta\mathbf{f}^{n}_{m+1} + \Delta\mathbf{W}^{n}_{m+1}\right] - (\mathbf{K}_T)^{n}_{m+1}\,\Delta\mathbf{w}^{n}_{m+1} = \mathbf{0}, \tag{7}$$

where the tangent stiffness matrix $(\mathbf{K}_T)^{n}_{m+1}$ is defined as

$$(\mathbf{K}_T)^{n}_{m+1} = \frac{\partial(\Delta\boldsymbol{\Psi}^{n})}{\partial(\Delta\mathbf{w}^{n})} = \left[(\mathbf{K}^{e}_{e})^{n} + (\mathbf{K}_f)^{n}\right]. \tag{8}$$

The first term in (8) represents the elastic stiffness matrix of the plate elements, while the second one corresponds to the stiffness matrix of the foundation. The term $\Delta\mathbf{w}^{n}_{m+1}$ from equation (7) can be used to find the accumulated displacement vector after $(n+1)$ iterations in the $(m+1)$ load increment acording to the following equation

$$\Delta\mathbf{w}^{n+1}_{m+1} = \mathbf{w}^{n}_{m+1} + \Delta\mathbf{w}^{n}_{m+1}. \tag{9}$$

At the end every iteration there should be a convergence check in order to see whether the iterative solution converged within the out-of-balance force tolerance (see equation (7)).

S t r e s z c z e n i e

W pracy zaprezentowano obliczenia za pomocą metody elementów skończonych płyty grubej spoczywającej na dwuparametrowym podłożu wielowarstwowym. Model numeryczny pozwala na dodanie dowolnej liczby warstw podłoża. Wyrażenia na obliczenie macierzy sztywności podłoża bazują na wykorzystaniu 18-węzłowego międzypowierzchniowego elementu kontaktowego o zerowej grubości. W celu zamodelowania elementu płyty grubej wykorzystano 9-węzłowy element płytowy z rodziny Lagrange'a zdefiniowany według teorii Mindlina. Sformułowanie uwzględnia zarówno wpływ naprężeń ścinających w płycie jak również warunki kontaktu jednostronnego pomiędzy płytą a podłożem. Warunki kontaktu jednostronnego są zrealizowane przez usunięcie odpowiedniej części macierzy sztywności podłoża, która przypisana jest do węzłów nie będących w kontacie. Poprawność zaproponowanego algorytmu zilustrowano kilkoma przykładmi numerycznymi.

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