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Necessary and sufficient conditions for the minimum energy control of positive discrete-time linear systems with bounded inputs

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Abstract. The minimum energy control problem for the positive discrete-time linear systems with bounded inputs is formulated and solved. Necessary and sufficient conditions for the existence of solution to the problem are established. A procedure for solving of the problem is proposed and illustrated by a numerical example.

Key words: positive, discrete-time, minimum energy control, bounded inputs, procedure.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [1, 2]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc. Positive linear systems consisting of n subsystems with different fractional orders have been analyzed in [3].

The minimum energy control problem for standard linear systems has been formulated and solved by J. Klamka [4–7] and for 2D linear systems with variable coefficients in [8]. The controllability and minimum energy control problem of fractional discrete-time linear systems has been investigated by Klamka in [7]. The minimum energy control of positive continuous-time linear systems has been addressed in [9, 10]. The minimum energy control of positive fractional linear systems has been considered in [9, 10] and of descriptor positive systems in [11, 12].The minimum energy control of positive continuous-time linear systems with bounded inputs has been addressed in [13] and of discrete-time linear systems with bounded inputs in [14, 15].

In this paper the minimum energy control problem for positive discrete-time linear systems with bounded inputs will be formulated and solved.

The paper is organized as follows. In Sec. 2 the basic definitions and theorems of the positive discrete-time linear systems are recalled and the necessary and sufficient conditions for the reachability of the positive systems are given. The minimum energy control problem of the positive linear systems with bounded inputs is formulated and solved in Sec. 3. Necessary and sufficient conditions for the existence of solution of the problem are established and a procedure for computation of the optimal inputs and the minimum value of the performance index are also presented. Concluding remarks are given in Sec. 4.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n \times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n \times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n \times n$ identity matrix.

2. Preliminaries and the problem formulation

Consider the discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i, \tag{1}$$

where $x_i \in \Re^n$ and $u_i \in \Re^m$ are the state and input vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$.

Definition 1. [1, 2] The system (1) is called the internally positive if $x_i \in \Re_+^n$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.

Theorem 1. [1, 2] The system (1) is internally positive if and only if

$$A \in \Re_+^{n\times n}, \quad B \in \Re_+^{n\times m}. \tag{2}$$

Definition 2. The positive system (1) (or the positive pair (A, B)) is called reachable in q steps if for any given final state $x_f \in \Re_+^n$ there exists an input sequence $u_k \in \Re_+^m$, for $k = 0, 1, \ldots, q-1$ that steers the state of the system from zero initial state $x_0 = 0$ to the state x_f, i.e. $x_q = x_f$.

A real square matrix is called monomial if each its row and each its column contains only one positive entry and the remaining entries are zero.

Theorem 2. [2] The positive system (1) is reachable in q steps if and only if the reachability matrix

$$R_q = [\ B \quad AB \quad \ldots \quad A^{q-1}B\ ]. \tag{3}$$

contains n linearly independent monomial columns.

For single input systems ($m = 1$) $q = n$ the positive system (1) is reachable in n steps if and only if the reachability matrix R_n is a monomial matrix. In this case there exists only

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one input sequence $u_k \in \Re_+^m$, $k = 0, 1, \ldots, n-1$ that steers the state of the system from $x_0 = 0$ to the state $x_f \in \Re_+^n$ given by

$$\begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_0 \end{bmatrix} = R_n^{-1} x_f. \tag{4}$$

If $m > 1$ and the positive system (1) is reachable in q steps then there exist many input sequences $u_k \in \Re_+^m$, $k = 0, 1, \ldots, q-1$ that steers the state of the system from $x_0 = 0$ to the state $x_f \in \Re_+^n$. Among these inputs sequences we are looking for the sequence $u_k \in \Re_+^m$, $k = 0, 1, \ldots, n-1$ that minimizes the performance index

$$I(u) = \sum_{k=0}^{q-1} u_k^T Q u_k, \tag{5}$$

where $Q \in \Re_+^{m \times m}$ is a symmetric positive defined matrix.

The minimum energy control problem for the positive discrete-time linear systems (1) with bounded inputs can be stated as follows: Given the matrices (2), the final state $x_f \in \Re_+^n$ and matrix $Q$of the performance index (5), find an input sequence $u_k \in \Re_+^m$, $k = 0, 1, \ldots, q-1$ satisfying the condition

$$u_k < U (U \in \Re_+^m \text{ is given}) \text{ for } k = 0, 1, \ldots, q-1, \tag{6}$$

that steers the state vector of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (5).

3. Problem solution

To solve the problem we define the matrix

$$W_q = R_q Q_q^{-1} R_q^T \in \Re^{n \times n}, \tag{7}$$

where R_q is defined by (3), T denote the transpose and

$$Q_q^{-1} = \text{blockdiag}[Q^{-1}, ..., Q^{-1}] \in \Re_+^{qm \times qm}. \tag{8}$$

Note that the condition (8) is met if and only if the matrix Q is diagonal.

Remark 1. If all columns of the matrix (3) are monomial and the matrix Q is diagonal then the matrix (7) is also diagonal.

Remark 2. It is easy to check that all columns of the matrix (3) are monomial if and only if the pair (A, B) is reachable and all columns of the matrix $[A\ B]$ are monomial.

Lemma 1. The matrix (7) is diagonal if and only if the pair (A, B) is reachable and all columns of the matrix

$$R_{n+1} = [\ B \quad AB \quad \ldots \quad A^n B\ ] \tag{9}$$

are monomial and the matrix Q is diagonal.

Proof. By Theorem 2 the pair (A, B) is reachable if and only if n columns of the matrix (9) are monomial. If all columns of the matrix (9) are monomial then from the equality $A^k B = A(A^{k-1}B)$, $k = 1, 2, \ldots, q$ it follows that all columns of the matrix (8) are monomial for any $q = n+1, \ldots$.

By Remark 1 if all columns of the matrix (3) for any q are monomial and the matrix Q is diagonal then the matrix (7) is diagonal for any $q = 1, 2, \ldots$.

Lemma 2. If the pair (A, B) is reachable and all columns of the matrix (9) are monomial and the matrix Q is diagonal then

$$W_q^{-1} x_f \in \Re_+^n \tag{10}$$

for any $x_f \in \Re_+^n$.

Proof. By Lemma 1 if the assumptions are satisfied then the matrix (7) is diagonal and $W_q^{-1} \in \Re_+^{n \times n}$ since the diagonal entries of W_q are positive. Therefore, the condition (10) is met for any $x_f \in \Re_+^n$.

Lemma 3. If the pair (A, B) is reachable and all columns of the matrix (9) are monomial and the matrix Q is diagonal then the input sequence

$$\widehat{u}_q = \begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_0 \end{bmatrix} = Q_q^{-1} R_q^T W_q^{-1} x_f \in \Re_+^{qm} \tag{11}$$

steers the positive system from $x_0 = 0$ to $x_f \in \Re_+^n$.

Proof. Using the solution

$$x_i = A^i x_0 + \sum_{j=0}^{i-1} A^{i-j-1} B u_j \tag{12}$$

of the Eq. (1) for $x_0 = 0$ and $i = q$ and (11) we obtain

$$x_q = R_q \widehat{u}_q = R_q Q_q^{-1} R_q^T W_q^{-1} x_f = x_f \tag{13}$$

since (7) holds.

Lemma 4. If the diagonal matrix Q is scalar matrix

$$Q = \text{diag}[q_1, ..., q_1] \in \Re_+^{m \times m} \tag{14}$$

then the input sequence (11) is independent of Q and is given by

$$\widehat{u}_q = R_q^T [R_q R_q^T]^{-1} x_f \in \Re_+^{qm} \tag{15}$$

for any $x_f \in \Re_+^n$.

Proof. If (14) holds then from (7) we have

$$W_q = \frac{1}{q_1} R_q R_q^T \in \Re_+^{n \times n} \tag{16}$$

and

$$W_q^{-1} = q_1 [R_q R_q^T]^{-1} \in \Re_+^{n \times n}. \tag{17}$$

In this case the input sequence (11) is given by

$$\begin{aligned} \widehat{u}_q &= Q_q^{-1} R_q^T W_q^{-1} x_f \\ &= \frac{1}{q_1} R_q^T q_1 [R_q R_q^T]^{-1} x_f = R_q^T [R_q R_q^T]^{-1} x_f \end{aligned} \tag{18}$$

for any $x_f \in \Re_+^n$.

Theorem 3. Let the positive system (1) be reachable in q steps, the matrix $Q \in \Re_+^{m \times m}$ be diagonal and all columns of the matrix (9) be monomial. Let $\overline{u}_k \in \Re_+^m$, $k = 0, 1, \ldots, q-1$ be an input sequence satisfying (6) that steers the state of the

positive system (1) from $x_0 = 0$ to $x_f \in \Re_+^n$. Then the input sequence (11) satisfying (6) also steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (5), i.e. $I(\widehat{u}) \leq I(\overline{u})$. The minimal value of the performance index (5) is given by

$$I(\widehat{u}) = x_f^T W^{-1} x_f. \tag{19}$$

Proof is similar to the proof in [14].

Remark 3. If U in (6) decreases then the number q of steps needed to transfer the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ increases.

Therefore, the following theorem has been proved.

Theorem 4. There exists input sequence (11) that steers the state of the system (1) from $x_0 = 0$ to any given $x_f \in \Re_+^n$ and minimize the performance index (5) for diagonal matrix $Q \in \Re_+^{m\times m}$ and any $x_f \in \Re_+^n$ if and only If the system is reachable in q steps and all columns of the matrix (9) are monomial.

Proof. Necessity. For any given $x_f \in \Re_+^n$ the condition (10) is satisfied only if $W_q^{-1} \in \Re_+^{n\times n}$ and this holds if and only if $W_q \in \Re_+^{n\times n}$ is monomial matrix. Note that $Q_q^{-1} \in \Re_+^{qm\times qm}$ if and only if $Q \in \Re_+^{m\times m}$ is diagonal matrix. From (7) it follows that for diagonal matrix Q the matrix $W_q \in \Re_+^{n\times n}$ is diagonal if and only if all columns of the reachability matrix (9) are monomial.

Proof of sufficiency is similar to the proof of Theorem 3.

Theorem 5. If the assumptions of Theorem 4 are met then the minimum energy control problem with bounded inputs has a solution for arbitrary U in (6).

Proof. We shall show that if the number q in (3) increases then U in (6) decreases. Without loss of generality we can assume that the matrices A, B ($m = 1$) and Q have the forms

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & a_n \\ a_1 & 0 & \dots & 0 & 0 \\ 0 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{n-1} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} b \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad Q = [q_1]. \tag{20}$$

Using (3) for $q > n+1$, (7) and (20) we obtain

$$R_q = [\ B \quad AB \quad \dots \quad A^{q-1}B\ ] = \begin{bmatrix} b & 0 & 0 & \dots & 0 & a_1 \dots a_n b & 0 & \dots \\ 0 & a_1 b & 0 & \dots & 0 & 0 & a_1^2 a_2 \dots a_n b & \dots \\ 0 & 0 & a_1 a_2 b & \dots & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_1 \dots a_{n-1} b & 0 & 0 & \dots \end{bmatrix} \tag{21}$$

and

$$W_q = R_q Q_q^{-1} R_q^T = \frac{1}{q_1} \text{diag}[b^2 + (a_1 \dots a_n b)^2 + \dots, (a_1 b)^2 + (a_1^2 a_2 \dots a_n b)^2 + \dots, \dots, (a_1 \dots a_{n-1} b)^2 + \dots] \in \Re^{n\times n}. \tag{22}$$

From (11) and (22) we have

$$\widehat{u}_q = Q_q^{-1} R_q^T W_q^{-1} x_f = \frac{1}{q_1} \begin{bmatrix} b & 0 & 0 & \dots & 0 \\ 0 & a_1 b & 0 & \dots & 0 \\ 0 & 0 & a_1 a_2 b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_1 \dots a_n b \\ a_1 \dots a_n b & 0 & 0 & \dots & 0 \\ 0 & a_1^2 a_2 \dots a_n b & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \end{bmatrix} \tag{23}$$

$$\times q_1 \text{diag}\left[\frac{1}{b^2 + (a_1 \dots a_n b)^2 + \dots}, \frac{1}{(a_1 b)^2 + (a_1^2 a_2 \dots a_n b)^2 + \dots}, \dots, \frac{1}{(a_1 \dots a_{n-1} b)^2}\right] x_f$$

$$= \text{diag}\left[\frac{b}{b^2 + (a_1 \dots a_n b)^2 + \dots}, \frac{a_1 b}{(a_1 b)^2 + (a_1^2 a_2 \dots a_n b)^2 + \dots}, \dots\right] x_f.$$

From (23) if follows that if q increases then the components of $\widehat{u}_q$ decrease and for any given U in (6) there exists a number q for which the condition (6) is satisfied. Therefore, if the assumptions of Theorem 4 are met then the minimum energy control problem with bounded inputs has a solution for arbitrary U in (6).

The optimal input sequence (11) and the minimal value of the performance index (19) can be computed by the use of the following procedure.

Procedure 1.

Step 1. Knowing the matrices A, B, Q and using (3) and (7) compute the matrices R_q and W_q for a chosen q such that the matrix R_q contains at least $n$linearly independent monomial columns.

Step 2. Using (11) find the input sequence $u_k \in \Re_+^m$, $k = 0, 1, \ldots, q-1$ satisfying the condition (6). If the condition (6) is not satisfied increase q by one and repeat the computation for q + 1. If the matrix W_q is diagonal after some number of steps we obtain the desired input sequence satisfying the condition (6).

Step 3. Using (19) compute the minimal value of the performance index $I(\widehat{u})$.

Example 1. Consider the positive discrete-time linear system (1) with matrices

$$A = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{24}$$

and the performance index (5) with $Q = [2]$.

Find the input sequence $u_k \in \Re_+^m$, $k = 0, 1, \ldots$ satisfying the condition (6) with

$$u_k < \frac{1}{3}, \qquad k = 0, 1, \ldots \tag{25}$$

that steers the state of the system from zero state to final state $x_f = [\ 1 \quad 1\ ]^T \in \Re_+^2$ and minimize the performance index.

Note that in this case all columns of the reachability matrix

$$\begin{aligned} R_q &= [\ B \quad AB \quad \ldots \quad A^{q-1}B\ ] \\ &= \begin{bmatrix} 0 & 3 & 0 & 18 & \ldots \\ 1 & 0 & 6 & 0 & \end{bmatrix} \end{aligned} \tag{26}$$

are monomial.

Using the Procedure 1 we obtain the following:

Step 1. Using (7) and (15) we obtain

$$\begin{aligned} W_q = R_q Q_q^{-1} R_q^T &= \begin{bmatrix} 0 & 3 & 0 & 18 & \ldots \\ 1 & 0 & 6 & 0 & \end{bmatrix} \\ &\quad \operatorname{diag}[0.5, 0.5, 0.5, \ldots] \begin{bmatrix} 0 & 1 \\ 3 & 0 \\ 0 & 6 \\ 18 & 0 \\ \vdots & \end{bmatrix} \\ &= \begin{bmatrix} 9 + 18^2 + \ldots & 0 \\ 0 & 1 + 6^2 + \ldots \end{bmatrix} \in \Re_+^{2\times 2}. \end{aligned} \tag{27}$$

Step 2. Using (11), (26) and (27) we obtain

$$\begin{aligned} \widehat{u}_2 &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = Q_2^{-1} R_2^T W_2^{-1} x_f \\ &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{9} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ \frac{1}{3} \end{bmatrix}. \end{aligned} \tag{28a}$$

This input sequence does not satisfy the condition (25) and we compute

$$\begin{aligned} \widehat{u}_3 &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = Q_3^{-1} R_3^T W_3^{-1} x_f \\ &= \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} \frac{2}{9} & 0 \\ 0 & \frac{2}{37} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{37} \\ \frac{1}{3} \\ \frac{6}{37} \end{bmatrix}. \end{aligned} \tag{28b}$$

The input sequence (28b) also does not satisfy the condition (25) and we continue the computation

$$\begin{aligned} \widehat{u}_4 &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = Q_4^{-1} R_4^T W_4^{-1} x_f \\ &= \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 0 \\ 0 & 6 \\ 18 & 0 \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} \frac{2}{9+18^2} & 0 \\ 0 & \frac{2}{37} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{37} \\ \frac{3}{9+18^2} \\ \frac{6}{37} \\ \frac{18}{9+18^2} \end{bmatrix}. \end{aligned} \tag{28c}$$

The input sequence (28c) satisfies the condition (25) and by Theorem 3 is the optimal one that steers the state of the system in 4-steps from zero state to final state $x_f = [\ 1 \quad 1\ ]^T$ and minimizes the performance index (5) for $Q = [2]$.

Step 3. The minimal value of the performance index (19) is equal to

$$I(\widehat{u}_4) = x_f^T W_4^{-1} x_f = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{9+18^2} & 0 \\ 0 & \frac{2}{37} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2}{9+18^2} + \frac{2}{37}. \tag{29}$$

4. Concluding remarks

The minimum energy control problem for the positive discrete-time linear systems with bounded inputs has been formulated and solved. Necessary and sufficient conditions for the existence of solution to the minimum energy control problem have been established (Theorem 4). It has been shown that if the positive system is reachable in q steps, all columns of the reachability matrix (9) are monomial then the minimum energy control problem has a solution for arbitrary U in (6) (Theorem 5). The procedure for computation of the optimal input sequence has been proposed and illustrated by a numerical example.

These considerations can be extended to fractional positive linear systems with bounded inputs [9, 14].

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