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10.2478/v10170-011-0002-6

Archives of Control Sciences
Volume 21(LVII), 2011
No. 4, *pages 363–372*

Parametric optimization of neutral linear system with two delays with P-controller

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In the paper the problem of parametric optimization of a linear neutral system with two delays accomplished with respect to the integral quadratic performance index is formulated and solved. A method of computing the value of performance index relies on determining such a Lyapunov functional defined on the state space that its value for the initial state is equal to that of performance index. In the paper the form of Lyapunov functional is assumed and a method of computing its coefficients is given. An example illustrating the application of the theory discussed is presented. It concerns a system with the P-controller designed to control a plant with two delays both retarded and neutral type. For such a system the value of the considered performance index is determined.

Key words: parametric optimization, Lyapunov functional, time delay system, neutral system

1. Introduction

Lyapunov quadratic functionals are used to test the stability of systems, in computation of the critical delay values for time delay systems, in computation of the exponential estimates for the solutions of time delay systems, in calculation of the robustness bounds for uncertain time delay systems, to calculation of a quadratic performance index of quality for the process of parametric optimization for time delay systems. We construct the Lyapunov functionals for the system with time delay with a given time derivative. For the first time such Lyapunov functional was introduced by Repin [13] for the case of retarded time delay linear systems with one delay. Repin [13] delivered also the procedure for determination of coefficients of the functional. Duda [1] used the Lyapunov functional, which was proposed by Repin, for the calculation of the value of a quadratic performance index of quality in the process of parametric optimization for systems with time delay of retarded type and extended the results to the case of neutral type time delay system in [2]. Duda [3] presented a method of determining the Lyapunov functional for linear dynamic system with two lumped retarded type time delays in the

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Received 3.04.2011.

general case with no-commensurate delays and presented a special case with commensurate delays in which the Lyapunov functional could be determined by solving of a set of ordinary differential equations. Duda [4] presented a method of determining the Lyapunov functional for linear dynamic system with two delays both retarded and neutral type time delay and in the paper [5] presented a method of determining the Lyapunov quadratic functional for linear time-invariant system with k-non- commensurate neutral type time delays. Infante and Castelan [7] construction of the Lyapunov functional is based on a solution of a matrix differential-difference equation on a finite time interval. This solution satisfies symmetry and boundary conditions. Kharitonov and Zhabko [11] extended the results of Infante and Castelan [7] and proposed a procedure of construction of quadratic functionals for linear retarded type delay systems which could be used for the robust stability analysis of time delay systems. This functional was expressed by means of Lyapunov matrix, which depended on the fundamental matrix of time delay system. Kharitonov [8] extended some basic results obtained for the case of retarded type time delay systems to the case of neutral type time delay systems, and in [9] to the neutral type time delay systems with discrete and distributed delay. Kharitonov and Plischke [10] formulated the necessary and sufficient conditions for the existence and uniqueness of the delay Lyapunov matrix for the case of retarded system with one delay.

The paper presents the parametric optimization problem for a system with two delays both retarded and neutral type time delay with P-controller. To the best of author's knowledge, such parametric optimization problem has not been reported in the literature. An example illustrating the method is also presented.

2. Formulation of the parametric optimization problem

Let us consider the linear system with two delays both retarded and neutral type, whose dynamics is described by equations

$$\begin{cases} \frac{dx(t)}{dt} - D\frac{dx(t-\tau)}{dt} = Ax(t) + Bx(t-\tau) + Cu(t-r) \\ x(t_0) = x_0 \\ x(t_0+\theta) = \Phi(\theta) \\ u(t) = -Kx(t) \end{cases} \tag{1}$$

$$t \geqslant t_0,\ \theta \in [-r,0), \quad r \geqslant \tau > 0$$

$$A, B, D \in \mathbb{R}^{n\times n}, \quad x(t) \in \mathbb{R}^n, \quad \Phi \in W^{1,2}([-r,0),\mathbb{R}^n)$$

$$C \in \mathbb{R}^{n\times m}, K \in \mathbb{R}^{m\times n}, \quad u(t) \in \mathbb{R}^m$$

$W^{1,2}([-r,0),\mathbb{R}^n)$ is a space of all absolutely continuous functions defined on the interval $[-r,0)$ with values in $\mathbb{R}^n$ which derivative is in $L^2([-r,0),\mathbb{R}^n)$ -a space of a Lebesgue square integrable functions on interval $[-r,0)$ with values in $\mathbb{R}^n$.

We can reshape the equation (1) to the form

$$\begin{cases} \frac{dx(t)}{dt} - D\frac{dx(t-\tau)}{dt} = Ax(t) + Bx(t-\tau) - CKx(t-r) \\ x(t_0) = x_0 \\ x(t_0+\theta) = \Phi(\theta) \end{cases} \tag{2}$$

We introduce a new function y, defined by term

$$y(t) = x(t) - Dx(t-\tau) \quad for\, t \geqslant t_0 \tag{3}$$

Thus the equation (2) takes a form

$$\begin{cases} \frac{dy(t)}{dt} = Ay(t) + (AD+B)x(t-\tau) - CKx(t-r) \\ y(t) = x(t) - Dx(t-\tau) \\ y(t_0) = x_0 - D\Phi(-\tau) \\ x(t_0+\theta) = \Phi(\theta) \end{cases} \tag{4}$$

The state of the system (4) is a vector

$$S(t) = \begin{bmatrix} y(t) \\ x_t \end{bmatrix} \qquad \text{for} \quad t \geqslant t_0 \tag{5}$$

$$\text{where} \quad x_t \in W^{1,2}([-r,0),\mathbb{R}^n) \qquad x_t(\theta) = x(t+\theta)\; for\, \theta \in [-r,0)$$

The state space is defined by the formula

$$X = \mathbb{R}^n \times W^{1,2}([-r,0),\mathbb{R}^n) \tag{6}$$

We search for such matrix K whose minimizes the integral quadratic performance index

$$J = \int_{t_0}^{\infty} y^T(t)y(t)dt \tag{7}$$

3. The method of determining the value of the performance index

On the state space X we define a Lyapunov functional, positively defined, differentiable, with the derivative computed on the trajectory of the system (4) being negatively defined.

$$V(S(t)) = y^T(t)\alpha y(t) + \int_{-r}^{0} y^T(t)\beta(\theta)x(t+\theta)d\theta + \tag{8}$$

$$+\int_{-r}^{0}\int_{\theta}^{0} x^T(t+\theta)\delta(\theta,\sigma)x(t+\sigma)d\sigma d\theta$$

for $t \geqslant t_0$ where $\alpha = \alpha^T \in \mathbb{R}^{n\times n}$, $\beta \in C^1([-r,0],\mathbb{R}^{n\times n})$, $\delta \in C^1(\Omega,\mathbb{R}^{n\times n})$, $\Omega = \{(\theta,\varsigma) : \theta \in [-r,0], \varsigma \in [\theta,0]\}$. C^1 is a space of continuous functions with continuous derivative.

We identify the coefficients of the functional (8), assuming that its derivative computed on the trajectory of the system (4) satisfies the relationship

$$\frac{dV(S(t))}{dt} = -y^T(t)y(t) \qquad for\, t \geqslant t_0 \tag{9}$$

When the relationship (9) holds , we can easily determine the value of a square indicator of the quality of the parametric optimization, because

$$J = \int_{t_0}^{\infty} y^T(t)y(t)dt = V(S(t_0)) \tag{10}$$

4. Determination the coefficients of the functional (8)

We compute the derivative of the functional (8) on the trajectory of the system (4) according to the formula

$$\frac{dV(S(t))}{dt} = grad(V(S(t))\frac{dS(t)}{dt} \qquad for\, t \geqslant t_0 \tag{11}$$

The derivative of the functional (8), calculated on the basis of the formula (11), is given by the formula

$$\frac{dV(S(t))}{dt} = y^T(t)\left[A^T\alpha + \alpha A + \frac{\beta(0)+\beta^T(0)}{2}\right]y(t)+$$

$$+y^T(t)\left[2\alpha(B+AD)+\beta(0)D\right]x(t-\tau) + y^T(t)\left[-2\alpha CK - \beta(-r)\right]x(t-r)+$$

$$+\int_{-r}^{0} y^T(t)\left[A^T\beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta^T(\theta,0)\right]x(t+\theta)d\theta+$$

$$+\int_{-r}^{0} x^T(t-\tau)[(B+AD)^T\beta(\theta)+D^T\delta^T(\theta,0)]x(t+\theta)d\theta+$$

$$+\int_{-r}^{0} x^T(t-r)\left[-K^TC^T\beta(\theta)-\delta(-r,\theta)\right]x(t+\theta)d\theta+$$

$$-\int_{-r}^{0}\int_{\theta}^{0} x^T(t+\theta)\left[\frac{\partial\delta(\theta,\sigma)}{\partial\theta}+\frac{\partial\delta(\theta,\sigma)}{\partial\sigma}\right]x(t+\sigma)d\sigma d\theta \tag{12}$$

From equations (12) and (9) we obtain the system of equations (13) to (19)

$$A^T\alpha+\alpha A+\frac{\beta(0)+\beta^T(0)}{2}=-I \tag{13}$$

$$2\alpha(B+AD)+\beta(0)D=0 \tag{14}$$

$$-2\alpha CK-\beta(-r)=0 \tag{15}$$

$$A^T\beta(\theta)-\frac{d\beta(\theta)}{d\theta}+\delta^T(\theta,0)=0 \tag{16}$$

$$(B+AD)^T\beta(\theta)+D^T\delta^T(\theta,0)=0 \tag{17}$$

$$-K^TC^T\beta(\theta)-\delta(-r,\theta)=0 \tag{18}$$

$$\frac{\partial\delta(\theta,\sigma)}{\partial\theta}+\frac{\partial\delta(\theta,\sigma)}{\partial\sigma}=0 \tag{19}$$

for $\theta\in[-r,0]$, $\sigma\in[-r,0]$.

From equation (14) it results that

$$\beta(0)=-2\alpha(B+AD)D^{-1} \tag{20}$$

We put (20) into (13). After some calculations we get

$$\alpha P+P^T\alpha=I \tag{21}$$

where

$$P=BD^{-1} \tag{22}$$

From equation (21) we obtain the matrix α.

From equation (17) we obtain

$$\delta^T(\theta,0) = -A^T\beta(\theta) - P^T\beta(\theta) \tag{23}$$

We put (23) into (16). We get

$$\frac{d\beta(\theta)}{d\theta} = -P^T\beta(\theta) \tag{24}$$

We get the initial condition for the differential equation (24) from the formula (15)

$$\beta(-r) = -2\alpha CK \tag{25}$$

The solution of the differential equation (24) with the initial condition (25) is given by formula

$$\beta(\theta) = -2\exp\left(-P^T(\theta+r)\right)\alpha CK \tag{26}$$

for $\theta \in [-r,0]$.

The solution of the equation (19) is as below

$$\delta(\theta,\sigma) = \varphi(\theta-\sigma) \tag{27}$$

where $\varphi \in C^1\left([-r,r],\mathbb{R}^{n\times n}\right)$.

From equations (27) and (18) we obtain

$$\delta(-r,\theta) = \varphi(-r-\theta) = -K^TC^T\beta(\theta) \tag{28}$$

for $\theta \in [-r,0]$.

Hence

$$\varphi(\xi) = -K^TC^T\beta(-\xi-r) \tag{29}$$

for $\xi \in [-r,0]$.

Taking into account (27) and (29), we get the formula

$$\delta(\theta,\sigma) = -K^TC^T\beta(\sigma-\theta-r) \tag{30}$$

Upon taking the relation (26) into account, we get the formula

$$\delta(\theta,\sigma) = 2K^TC^T\exp\left(-P^T(\sigma-\theta)\right)\alpha CK \tag{31}$$

In this way we obtained all parameters of the Lyapunov functional (8).

5. Determination the value of the performance index

According to the formula (10) the value of the performance index is given by the formula

$$J = V\left(y(t_0), \Phi\right) = y^T(t_0)\alpha y(t_0) + \tag{32}$$

$$+\int_{-r}^{0} y^T(t_0)\beta(\theta)\Phi(\theta)d\theta + \int_{-r}^{0}\int_{\theta}^{0} \Phi^T(\theta)\delta(\theta,\sigma)\Phi(\sigma)d\sigma d\theta$$

We put the relations (26) and (31) into (32) and we get

$$J = y^T(t_0)\alpha y(t_0) - 2\int_{-r}^{0} y^T(t_0)\exp\left(-P^T(\theta+r)\right)\alpha CK\Phi(\theta)d\theta + \tag{33}$$

$$+2\int_{-r}^{0}\int_{\theta}^{0} \Phi^T(\theta)K^T C^T \exp\left(-P^T(\sigma-\theta)\right)\alpha CK\Phi(\sigma)d\sigma d\theta$$

6. Example

Let us consider the system described by equation

$$\begin{cases} \frac{dx(t)}{dt} - d\frac{dx(t-\tau)}{dt} = ax(t) + bx(t-\tau) - ckx(t-r) \\ x(t_0) = x_0 \in \mathbb{R} \\ x(t_0+\theta) = \Phi(\theta) \end{cases} \tag{34}$$

$$t \geqslant t_0,\ x(t) \in \mathbb{R}, \quad \theta \in [-r, 0), \quad a, b, c, d, k \in \mathbb{R}, \quad r \geqslant \tau > 0$$

We can reshape the equation (34) to the form

$$\begin{cases} \frac{dy(t)}{dt} = ay(t) + (b+ad)x(t-\tau) - ckx(t-r) \\ y(t) = x(t) - dx(t-\tau) \\ y(t_0) = x_0 - d\Phi(-\tau) \\ x(t_0+\theta) = \Phi(\theta) \end{cases} \tag{35}$$

$$t \geqslant t_0,\ x(t) \in \mathbb{R}, \quad \theta \in [-r, 0), \quad a, b, c, d, k \in \mathbb{R}, \quad r \geqslant \tau > 0$$

We search for such parameter k whose minimizes the integral quadratic performance index

$$J = \int_{t_0}^{\infty} y^T(t)y(t)dt = V(y(t_0), \Phi) \tag{36}$$

The Lyapunov functional V is defined by the formula

$$V(S(t)) = \alpha y^2(t) + \int_{-r}^{0} y(t)\beta(\theta)x(t+\theta)d\theta + \int_{-r}^{0}\int_{\theta}^{0} x(t+\theta)\delta(\theta,\sigma)x(t+\sigma)d\sigma d\theta \tag{37}$$

where according to (21), α is given by formula

$$\alpha = \frac{1}{2p} = \frac{d}{2b} \tag{38}$$

According to (26), the coefficient β is given by formula

$$\beta(\theta) = -2\alpha ck \exp\left(-\frac{b(\theta+r)}{d}\right) = -\frac{cdk}{b}\exp\left(-\frac{b(\theta+r)}{d}\right) \tag{39}$$

According to (31), the element δ is given by formula

$$\delta(\theta,\sigma) = 2\alpha c^2k^2 \exp\left(-\frac{b(\sigma-\theta)}{d}\right) = \frac{c^2dk^2}{b}\exp\left(-\frac{b(\sigma-\theta)}{d}\right) \tag{40}$$

The value of the performance index is given by formula

$$J = \frac{d}{2b}y^2(t_0) - \frac{cdk}{b}y(t_0)\int_{-r}^{0}\Phi(\theta)\exp\left(-\frac{b(\theta+r)}{d}\right)d\theta + \tag{41}$$

$$+\frac{c^2dk^2}{b}\int_{-r}^{0}\int_{\theta}^{0}\Phi(\theta)\Phi(\sigma)\exp\left(-\frac{b(\sigma-\theta)}{d}\right)d\sigma d\theta$$

The performance index is a quadratic function with respect to the variable k.

To get the optimal value of the gain k we compute the derivative of the performance index with respect to k and we equal it to zero.

$$\frac{dJ(k)}{dk} = 0 \tag{42}$$

$$k_{opt} = \frac{y(t_0)}{2c}\frac{\int_{-r}^{0}\Phi(\theta)\exp\left(-\frac{b(\theta+r)}{d}\right)d\theta}{\int_{-r}^{0}\int_{\theta}^{0}\Phi(\theta)\Phi(\sigma)\exp\left(-\frac{b(\sigma-\theta)}{d}\right)d\sigma d\theta} \tag{43}$$

7. Conclusions

The paper presents the parametric optimization problem for a system with two delays both retarded and neutral type time delay with P-controller.A method of computing the value of performance index relies on determining such a Lyapunov functional defined on the state space that its value for the initial state is equal to that of performance index. In the paper the form of Lyapunov functional is assumed and a method of computing its coefficients is given.

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