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Checking of the positivity of descriptor linear systems with singular pencils

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A method for checking of the positivity of descriptor continuous-time and discrete-time linear systems with singular pencil is proposed. The method is based on elementary row and column operations on the matrices of descriptor systems. Necessary and sufficient conditions for the positivity of the descriptor systems are established.

Key words: checking, positivity, descriptor, linear, system, singular pencil

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in [6, 9, 10]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.. The Drazin inverse of matrix to analysis of linear algebraic-differential equations has been applied in [4, 5, 8]. Standard descriptor control systems have been addressed in [7, 11]. Positive descriptor linear systems have been analyzed in [1-3, 14]. A method based on shuffle algorithm for checking of the positivity of descriptor linear systems with regular pencil has been proposed in [13].

In this paper a method for checking of the positivity of descriptor linear systems with singular pencil will be proposed and necessary and sufficient conditions for the positivity of the systems will be established.

The paper is organized as follows. In section 2 basic definitions and theorems concerning positive standard linear systems are recalled and the problem formulation is given. The main result of the paper is presented in section 3 and 4. The proposed method for checking of the positivity of descriptor continuous-time linear systems with singular pencil is presented in section 3 and its extension for discrete-time systems in section 4. Concluding remarks are given in section 5.

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The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Preliminaries and the problem formulation

Consider the descriptor continuous-time linear system with singular pencil

$$E\dot{x}(t) = Ax(t) + Bu(i), \quad x(0) = x_0 \tag{1}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, are the state and input vectors and $E, A\in\Re^{q\times n}$, $B\in\Re^{q\times m}$, $\operatorname{rank} E \leqslant \min(q,n)$ and

$$\operatorname{rank}\,[Es-A] \leqslant \min(q,n) \ \text{ for some } \ s\in C \ \text{ (the field of complex numbers).} \tag{2}$$

Let U_{ad} be a set of all given admissible inputs $u\in\Re^m$ of the system (1). A set of all initial conditions $x_0\in\Re^n$ for which the equation (1) has a solution $x(t)$ for $u(t)\in U_{ad}$ is called the set of consistent initial conditions and is denoted by X_c^0. The set X_c^0 depends on the matrices E, A, B but also on $u(t)\in U_{ad}$ [11].

Theorem 1 [11] *The equation* (1) *has a solution $x(t)$ for all $u(t)\in U_{ad}$ and zero initial conditions if and only if*

$$\operatorname{rank}\,[Es-A] = \operatorname{rank}\,[Es-A, B] \ \textit{for some} \ s\in C. \tag{3}$$

Definition 1 *The descriptor system* (1) *is called (internally) positive if $x(t)\in\Re_+^n$, for every consistent nonnegative initial condition $x_0\in\Re_+^n$ and all admissible inputs $u(t)\in\Re_+^m$, $t\geqslant 0$.*

Consider the standard continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(i), \quad x(0) = x_0 \tag{4}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, are the state and input vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$. The system (4) is called (internally) positive if $x(t)\in\Re_+^n$, $t\geqslant 0$ for every nonnegative initial condition $x_0\in\Re_+^n$ and all inputs $u(t)\in\Re_+^m$, $t\geqslant 0$.

Theorem 2 *The standard continuous-time linear system* (4) *is positive if and only if*

$$A\in M_n \quad B\in\Re_+^{n\times m} \tag{5}$$

where M_n is the set of $n\times n$ Metzler matrices.

The following elementary row (column) operations will be used:

1. Multiplication of the ith row (column) by a real number c. This operation will be denoted by $L[i \times c]$ ($R[i \times c]$).
2. Addition to the ith row (column) of the jth row (column) multiplied by a real number c. This operation will be denoted by $L[i + j \times c]$ ($R[i + j \times c]$).
3. Interchange of the ith and jth rows (columns). This operation will be denoted by $L[i, j]$ ($R[i, j]$).

A method for checking the positivity of the descriptor linear system will be proposed. The method is based on elementary row and column operations.

3. Checking positivity of descriptor systems

Performing elementary row operations on the array

$$E \;\; A \;\; B \tag{6}$$

or equivalently on the equation (1) we obtain

$$\begin{array}{ccc} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{array} \tag{7}$$

and

$$E_1\dot{x}(t) = A_1x(t) + B_1u(t) \tag{8a}$$

$$0 = A_2x(t) + B_2u(t) \tag{8b}$$

where $E_1 \in \Re^{r\times n}$ has full row rank. If $\operatorname{rank} E_1 = r$ then there exists a nonsingular matrix $P \in \Re^{n\times n}$ of elementary column operations such that

$$E_1P = \begin{bmatrix} \bar{E}_1 & 0 \end{bmatrix}, \;\; \bar{E}_1 \in \Re^{r\times r}, \;\; 0 \in \Re^{r\times(n-r)}, \;\; \det \bar{E}_1 \neq 0. \tag{9}$$

Defining the new state vector

$$\bar{x}(t) = P^{-1}x(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix}, \quad \bar{x}_1(t) \in \Re^{r}, \quad \bar{x}_2(t) \in \Re^{(n-r)} \tag{10}$$

and using (9) we may write the equations (8) in the form

$$E_1PP^{-1}\dot{x}(t) = \bar{E}_1\dot{\bar{x}}_1(t) = A_1PP^{-1}x(t) + B_1u(t) = \bar{A}_{11}\bar{x}_1(t) + \bar{A}_{12}\bar{x}_2(t) + B_1u(t) \tag{11a}$$

$$0 = A_2PP^{-1}x(t) + B_2u(t) = \bar{A}_{21}\bar{x}_1(t) + \bar{A}_{22}\bar{x}_2(t) + B_2u(t) \tag{11b}$$

where

$$A_1P = [\ \bar{A}_{11} \quad \bar{A}_{12}\ ], \quad \bar{A}_{11} \in \Re^{r\times r}, \quad \bar{A}_{12} \in \Re^{r\times(n-r)},$$
$$A_2P = [\ \bar{A}_{21} \quad \bar{A}_{22}\ ], \quad \bar{A}_{21} \in \Re^{(q-r)\times r}, \quad \bar{A}_{22} \in \Re^{(q-r)\times(n-r)}. \tag{11c}$$

Case 1.

If

$$q > n \quad \text{and} \quad \text{rank}\bar{A}_{22} = n - r \tag{12}$$

then performing elementary row operations on the array

$$\bar{A}_{21} \quad \bar{A}_{22} \quad \bar{B}_2 \tag{13}$$

or equivalently on (11b) we obtain

$$\begin{matrix} \bar{A}_{31} & \bar{A}_{32} & \bar{B}_2 \\ \hat{A}_{31} & 0 & \hat{B}_2 \end{matrix} \tag{14}$$

and

$$0 = \bar{A}_{31}\bar{x}_1(t) + \bar{A}_{32}\bar{x}_2(t) + \bar{B}_2u(t) \tag{15a}$$
$$0 = \hat{A}_{31}\bar{x}_1(t) + \hat{B}_2u(t) \tag{15b}$$

where

$$\bar{A}_{31} \in \Re^{(n-r)\times r}, \quad \bar{A}_{32} \in \Re^{(n-r)\times(n-r)}, \quad \bar{B}_2 \in \Re^{(n-r)\times m}, \quad \det\bar{A}_{32} \neq 0$$
$$\hat{A}_{31} \in \Re^{(q-n)\times r}, \quad \hat{B}_2 \in \Re^{(q-n)\times m}. \tag{15c}$$

From (15a) we have

$$\bar{x}_2(t) = -\bar{A}_{32}^{-1}\bar{A}_{31}\bar{x}_1(t) - \bar{A}_{32}^{-1}\bar{B}_2u(t). \tag{16}$$

Substituting of (16) into (11a) yields

$$\bar{E}_1\dot{\bar{x}}_1(t) = (\bar{A}_{11} - \bar{A}_{12}\bar{A}_{32}^{-1}\bar{A}_{31})\bar{x}_1(t) + (B_1 - \bar{A}_{12}\bar{A}_{32}^{-1}\bar{B}_2)u(t) \tag{17}$$

and after premultiplying of (17) by $\bar{E}_1^{-1}$ we obtain

$$\dot{\bar{x}}_1(t) = \bar{A}_1\bar{x}_1(t) + \bar{B}_1u(t) \tag{18a}$$

where

$$\bar{A}_1 = \bar{E}_1^{-1}(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{32}^{-1}\bar{A}_{31}) \in \Re^{r\times r}, \quad \bar{B}_1 = \bar{E}_1^{-1}(B_1 - \bar{A}_{12}\bar{A}_{32}^{-1}\bar{B}_2) \in \Re^{r\times m}. \tag{18b}$$

From (15) it follows that the consistent initial condition $\bar{x}_0 = \begin{bmatrix} \bar{x}_{10} \\ \bar{x}_{20} \end{bmatrix}$ and the admissible inputs $u(t)$ should satisfy the conditions

$$0 = \bar{A}_{31}\bar{x}_{10} + \bar{A}_{32}\bar{x}_{20} + \bar{B}_2 u(0) \quad \text{and} \quad 0 = \hat{A}_{31}\bar{x}_{10} + \hat{B}_2 u(0). \tag{19}$$

Note that by (10) $\bar{x}_0 \in \Re^n_+$ for any $x_0 \in \Re^n_+$ if and only if $P^{-1} \in \Re^{n\times n}_+$.

<u>Case 2.</u>

If the condition (12) is not satisfied then from (11a) we have

$$\dot{\bar{x}}_1(t) = \hat{A}_1\bar{x}_1(t) + \hat{A}_2\bar{x}_2(t) + \hat{B}_1 u(t) \tag{20a}$$

where

$$\hat{A}_1 = \bar{E}_1^{-1}\bar{A}_{11}, \quad \hat{A}_2 = \bar{E}_1^{-1}\bar{A}_{12}, \quad \hat{B}_1 = \bar{E}_1^{-1}B_1. \tag{20b}$$

In this case the consistent initial condition $\bar{x}_0$ and the admissible inputs $u(t)$ should satisfy the conditions

$$0 = \bar{A}_{21}\bar{x}_{10} + \bar{A}_{22}\bar{x}_{20} + \bar{B}_2 u(0). \tag{21}$$

Note that choosing arbitrary $\bar{x}_2(t) \in \Re^{n-r}_+$, $t \geqslant 0$ from (20a) we can find

$$\bar{x}_1(t) = e^{\hat{A}_1 t}\bar{x}_{10} + \int_0^t e^{\hat{A}_1(t-\tau)}[\hat{A}_2\bar{x}_2(\tau) + \hat{B}_1 u(\tau)]d\tau \tag{22}$$

for given $\bar{x}_{10}$ and $u(t)$.

By Theorem 2 the system (18) is positive if and only if

$$\bar{A}_1 \in M_r, \quad \bar{B}_1 \in \Re^{r\times m}_+ \tag{23}$$

and the system (20) is positive if and only if

$$\hat{A}_1 \in M_r, \quad \hat{A}_2 \in \Re^{r\times(n-r)}_+, \quad \text{and} \quad \hat{B}_1 \in \Re^{r\times m}_+. \tag{24}$$

Therefore, the following theorem has been proved.

Theorem 3 *Let the descriptor system* (1) *satisfy the assumptions* (3). *The system satisfying the condition* (12) *and* $P^{-1} \in \Re^{n\times n}_+$ *is positive for nonnegative consistent initial condition and admissible inputs if and only if the condition* (23) *is met. If the condition* (12) *is not satisfied then the descriptor system is positive if and only if the conditions* (24) *are met and* $\bar{x}_2(t) \in \Re^{n-r}$, $t \geqslant 0$ *can be chosen arbitrarily so that* (11b) *holds.*

The presented method of checking of the positivity of descriptor systems (1) will be illustrated by the following two simple numerical examples.

Example 1 Consider the descriptor system (1) with the matrices

$$E=\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix},\quad A=\begin{bmatrix}-2&0\\0&1\\1&-1\end{bmatrix},\quad B=\begin{bmatrix}1\\0\\-1\end{bmatrix}. \tag{25}$$

In this case $q=3$, $n=2$, $m=1$, $\operatorname{rank}E=1$

$$\operatorname{rank}\,[Es-A]=\operatorname{rank}\begin{bmatrix}s+2&0\\0&-1\\-1&1\end{bmatrix}=2\quad\text{for all}\quad s\in C \tag{26}$$

and

$$E_1=[1\ \ 0],\ A_1=[-2\ \ 0],\ B_1=[1],\quad A_2=\begin{bmatrix}0&1\\1&-1\end{bmatrix},\quad B_2=\begin{bmatrix}0\\-1\end{bmatrix}. \tag{27}$$

The matrix $P=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $\bar{E}_1=[1], \bar{x}(t)=x(t)=\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$ and the equations (11) have the form

$$\dot{x}_1(t)=-2x_1(t)+u(t) \tag{28a}$$

$$\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}x_1(t)+\begin{bmatrix}1\\-1\end{bmatrix}x_2(t)+\begin{bmatrix}0\\-1\end{bmatrix}u(t). \tag{28b}$$

Note that the condition (12) is satisfied since $\operatorname{rank}\bar{A}_{22}=\operatorname{rank}\begin{bmatrix}1\\-1\end{bmatrix}=n-r=1$. Performing on the array

$$\begin{array}{ccc}\bar{A}_{21}&\bar{A}_{22}&B_2\end{array}=\begin{array}{ccc}0&1&0\\1&-1&-1\end{array} \tag{29}$$

the elementary row operation $L[2+1\times 1]$, we obtain

$$\begin{array}{ccc}\bar{A}_{31}&\bar{A}_{32}&\bar{B}_2\\\hat{A}_{31}&0&\hat{B}_2\end{array}=\begin{array}{ccc}0&1&0\\1&0&-1\end{array} \tag{30}$$

and

$$x_2(t)=0,\quad x_1(t)=u(t). \tag{31}$$

From (28a) and (31) we have $\dot{x}_1(t) = -x_1(t)$ and

$$x_1(t) = e^{-t}x_{10} = u(t). \tag{32}$$

Therefore, the consistent initial condition and admissible input should satisfy (32) and $x_{10} = u(0)$, $x_{20} = 0$.

Example 2 Check the positivity of the descriptor system (1) with the matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{33}$$

In this case $q = 2$, $n = 3$, $m = 1$, $\operatorname{rank} E = 1$

$$\operatorname{rank}[Es - A] = \operatorname{rank}\begin{bmatrix} s+2 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} = 2 \quad \text{for some} \quad s \in C \tag{34}$$

and

$$E_1 = [1 \;\; 0 \;\; 0], \quad A_1 = [-2 \;\; 0 \;\; 0], \quad B_1 = [1], \quad A_2 = [0 \;\; 1 \;\; -1], \quad B_2 = [-1]. \tag{35}$$

Note that in this case the condition (12) is not satisfied since $q = 2 < n = 3$. The matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\bar{E}_1 = [1]$, $\bar{x}(t) = x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ and the equations (20a) and (11b) have the forms

$$\dot{x}_1(t) = -2x_1(t) + u(t), \quad x_2(t) = x_3(t) + u(t). \tag{36}$$

From (36) it follows that the descriptor system with (33) is positive for any $x_3(t) > 0$ and $u(t) > 0$, $t > 0$ and consistent initial condition $x_{10} \geqslant 0$ and $x_{20} = x_{30} + u(0)$.

4. Extension to the descriptor discrete-time linear systems

The presented method can be easily extended to the descriptor discrete-time linear systems

$$Ex_{i+1} = Ax_i + Bu_i, \quad i \in Z_+ = \{0, 1, \ldots\} \tag{37}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$ are the state and input vectors and $E, A \in \Re^{q \times n}$, $B \in \Re^{q \times m}$. It is assumed that $\operatorname{rank} E \leqslant \min(q, n)$ and

$$\operatorname{rank}[Ez - A] \leqslant \min(q, n) \quad \text{for some} \quad z \in C \quad \text{(the field of complex number).} \tag{38}$$

The descriptor system (37) is called (internally) positive if $x_i \in \Re_+^n$, for every consistent nonnegative initial condition $x_0 \in \Re_+^n$ and all admissible inputs $u_i \in \Re_+^m$, $i \in Z_+$.

The standard discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i, \quad A \in \Re^{n\times n}, \quad B \in \Re^{n\times m}, \quad i \in Z_+ \tag{39}$$

is called (internally) positive if $x_i \in \Re^n_+$ for any $x_0 \in \Re^n_+$ and all inputs $u_i \in \Re^m_+$, $i \in Z_+$.

Theorem 4 [10] *standard discrete-time linear system* (37) *is positive if and only if*

$$A \in \Re^{n\times n}_+, \quad B \in \Re^{n\times m}_+. \tag{40}$$

Remark 1 From comparison of Theorem 2 and 4 it follows that for descriptor discrete-time systems (37) in Theorem 3 the conditions $\bar{A}_1 \in M_r$, $\hat{A}_1 \in M_r$ should be substituted by the conditions $\bar{A}_1 \in \Re^{r\times r}_+$ and $\hat{A}_1 \in \Re^{r\times r}_+$.

With slight modifications (see Remark 1) the considerations presented in section 3 can be extended to the discrete-time systems (37).

Example 3 Consider the descriptor discrete-time linear system (37) with the matrices

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}. \tag{41}$$

In this case $q = 3$, $n = 2$, $m = 1$, $\operatorname{rank} E = 1$

$$\operatorname{rank}\,[Ez - A] = \operatorname{rank} \begin{bmatrix} 0 & z-2 \\ -1 & 1 \\ -1 & -z+3 \end{bmatrix} = 2 \quad \text{for some} \quad z \in C. \tag{42}$$

Performing on the array

$$\begin{matrix} E & A & B \end{matrix} = \begin{matrix} 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & -3 & -2 \end{matrix} \tag{43}$$

the elementary row operation $L[3+1\times 1]$, we obtain

$$\begin{matrix} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{matrix} = \begin{matrix} 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{matrix} \tag{44}$$

and the equations

$$x_{2,i+1} = 2x_{2,i} + u_i, \quad x_{1,i} = x_{2,i} + u_i, \quad i \in Z_+. \tag{45}$$

where $x_i = \begin{bmatrix} x_{1,i} \\ x_{2,1} \end{bmatrix}$, $i \in Z_+$. From (45) it follows that the descriptor system is positive for any input $u_i \geqslant 0$, $i \in Z_+$ and the consistent initial condition $x_{10} = x_{20} + u_0$ and the arbitrary nonnegative x_{20}.

5. Concluding remarks

A method for checking of the positivity of descriptor continuous-time and discrete-time linear systems with singular pencil has been proposed. The method is based on elementary row and column operations on the matrices E, A and B of descriptor system. Necessary and sufficient conditions for the positivity of the descriptor systems have been established (Theorem 3). The effecteveness of the proposed method has been demonstrated on numerical examples. The method can be extended to positive descriptor 2D linear systems.

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