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Control of a planar robot in the flight phase using transverse function approach

D. PAZDERSKI* and K. KOZŁOWSKI

Chair of Control and Systems Engineering, Poznań University of Technology, 3a Piotrowo St., 60-965 Poznań, Poland

Abstract. This paper deals with stabilization and tracking control problems defined with respect to a planar mechanical structure similar to Raibert's robot. The proposed control solution is based on formal analysis of the control system on a Lie group. In order to take advantage of Lie group theory a dynamic extension of the robot kinematics is introduced. To cope with non-zero angular momentum the controller based on transverse functions is employed. Properties of the closed-loop control system are investigated based on simulations including practical stabilization at neighborhood of a constant point or a reference trajectory.

Key words: transverse functions, hopping robots, space manipulator, nonholonomic system, control motion task, systems on Lie groups.

1. Introduction

Control of nonholonomic systems has been an important research issue in robotics for a few decades. This is due to the fact that nonintegrable phase constraints can be observed for various robotic structures. Basically, one can distinguish two main sources of these constraints. The first one, commonly met in practice for a wide class of kinematic models of wheeled vehicles, arises due to interaction between the bodies during ideal rolling. The second one is a consequence of the fundamental laws of physics, which manifest themselves in the form of the conservation principle of angular momentum in the absence of external torques. An example of kinematic description of such systems can be found in [1].

In this paper we deal with a class of planar multi-body systems subject to phase constraints as a result of angular momentum conservation. These constraints reduce the degree of freedom of such mechanical systems that can hinder reconfiguration of their links. This issue can be observed for space robots in the permanent free flight. A more typical example of constrained systems are walking and hopping robots moving in the presence of gravity. These robots are governed by a hybrid dynamics which is defined at stance and flight phases as well it describes transitions stages between these phases, cf. [2, 3]. For these mechanisms the duration of flight phase is limited by assumption.

The control problem dedicated to robots with kinematic constraints coming from the angular momentum principle has been investigated in the robotics literature. Typically, the authors take advantage of open-loop motion planning supported by nonlinear optimization. Some propositions have been formulated for space manipulators [4]. By contrast, not many papers are devoted to analytical methods based on the closed-loop control. Worth particular mention are works by Xin *et al.* [5], Grizzle and others [6] and Rehman and Michalska [7].

This paper refers to the second group of the specified control methods. It proposes an alternative control solution based on the transverse function approach for a two-link flying mechanical structure similar to the Raibert's robot [8]. The main purpose of this works is to design the algorithm for the specific nonholonomic kinematics with a drift introduced by non-zero angular momentum.

The theory of transverse functions introduced by Morin and Samson [9] so far has been applied to various nonholonomic systems including simple wheeled robots [10], wheeled robots with trailers [11, 12], nonholonomic ball [13], trident-snake robot and others [14]. The crucial property of the controller based on transverse functions is its possibility to recover approximately non-feasible directions in the phase space of the given constrained system. To be more clear, the nonholonomic system can behave similarly to the holonomic one. As a result, it becomes possible to stabilize the system configuration in some vicinity of desired point or trajectory, even when a permanent disturbance occurs. Simultaneously, smoothness of the control law gives possibility to increase robustness of the closed-loop system to some class of disturbances (the robustness issue concerning control of nonholonomic systems have been reported in some papers [15–18]).

The proposed control design takes advantage of a Lie group theory which is obtained by a dynamical extension and integration of nilpotent control Lie algebra specified for the extended kinematics. Using Lie group allows proposing an almost globally defined control solution. Here we base on quite well established foundations of differential geometry, Lie algebras and Lie groups and their application in control theory. Basically, control methods taking advantage of differential geometry have been originated more than 30 years ago – cf. [19–22]. One of the first monograph publication that shows how to apply theses mathematical tools to robotic manipulators were written by Murray *et al.* [23]. In addition one

*e-mail: Dariusz.Pazderski@put.poznan.pl

can cite works done by Tchoń [24], Selig [25], Canudas de Wit *et al.* [26] and Bloch [27].

The main aim of this paper is to demonstrate possibility of using these techniques to cope with non-classic control problem defined in the precedence of any bounded and non-admissible drift which cannot be trivially compensated. According to the best authors' knowledge the control solution presented in this paper is the first proposal of control paradigm taking advantage of transverse functions addressed for nonholonomic systems in the flight phase.

The paper is organized as follows. In Sec. 2 the robot kinematics is considered and its fundamental properties are described using a Lie algebra. In order to make this work more clear selected issues concerning Lie groups and algebras are presented. The theory allows showing how the considered system can be described on a Lie group. Section 3 presents the following stages of the controller design. Firstly, a transverse function for the given control system is defined. Secondly, the tracking error is considered and the control law taking advantage of optimization of control signal effort is discussed. The results of extensive simulations for selected cases are presented in Sec. 4. The concluding remarks are given in Sec. 5.

2. Model

2.1. Kinematics. Consider a planar robot presented in Fig. 1 which is composed of two links connected by a revolute joint with rotation axis passing through point O. To simplify description, we refer to a structure of so-called Raibert's robot and make the following assumptions:

- position of point O corresponds to position of center of mass (COM) of link 1,
- link 2 is a translation joint equipped with point mass m moving along the line passing through point O,
- the robot in the flight phase is isolated, namely there is no interaction between the links and the environment.

The latter assumption is well satisfied in the space. However, it can be also justified for a hopping robot when it moves in the air – then the interaction forces and torques can be usually neglected for some short period of time.

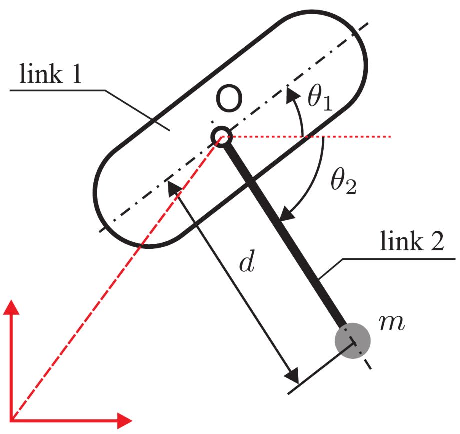

Fig. 1. Raibert's robot in the flight phase

Basically, the planar motion is a superposition of translation and rotation. In this paper we omit description of linear motion concerning translation of center of mass of the robot which is governed by Newton's laws and the gravity. Instead, we focus on rotation of the flying structure and its internal configuration.

Inertia of the first and second links is denoted by I_1 and I_2, respectively. The angles θ_1 and θ_2 describe orientation of the first and second link with respect to the inertial frame. It is assumed that values of these angles are positive for anticlockwise rotation (for the robot configuration presented in Fig. 1 $\theta_1 > 0$ while $\theta_2 < 0$). Notice that length $d > 0$ of link 2 can be adjusted that modifies inertia I_2. Applying Steiner's theorem one can easily conclude that

$$I_2 := md^2, \tag{1}$$

where $m > 0$.

Now, assume that angles θ_1 and θ_2 and distance d are configuration variables for the considered mechanical system. Then define configuration

$$q := [q_1\ q_2\ q_3]^\top := [\theta_1\ \theta_2\ d]^\top \in \mathcal{Q} \subset \mathbb{T}^2 \times \mathbb{R}_+. \tag{2}$$

Since the considered system is isolated one can refer to the angular momentum conservation principle. Calculating angular momentum of the considered mechanical structure with respect to its center of mass gives

$$\sigma = I_1\dot{\theta}_1 + I_2\dot{\theta}_2, \tag{3}$$

while $\sigma = \text{const}$. Then based on (2) and (1) the following Phaffian constraint can be defined

$$A(q)\dot{q} = \sigma, \tag{4}$$

where $A(q) := \left[I_1\ mq_3^2\ 0\right] \in \mathbb{R}^{1\times 3}$. Analyzing the phase constraint (4) one can prove that there is no function $\gamma(q) \in \mathbb{R}$, such that $\dfrac{\partial\gamma(q)}{\partial q} = A(q)$. Hence, it can be concluded that the constraint (4) is nonintegrable, which implies that the considered mechanism is a nonholonomic system.

Consequently, based on (4) one can consider the following kinematics

$$\dot{q} = \begin{bmatrix}0\\0\\1\end{bmatrix} u_1 + \begin{bmatrix}q_3^2\\-a\\0\end{bmatrix} u_2 + \begin{bmatrix}\eta_1\\\eta_2\\0\end{bmatrix} \sigma, \tag{5}$$

where $a := \dfrac{I_1}{m}$, u_1 and u_2 denote input signals, while η_1 and η_2 are real functions satisfying

$$I_1\eta_1 + mq_3^2\eta_2 - 1 = 0. \tag{6}$$

In the general case, when the angular momentum σ is nonzero, Eq. (5) describes the nonholonomic kinematics with the drift.

Interpreting the kinematic control inputs u_1 and u_2 one can conclude that u_2 is rather a theoretical signal not well suited to control the robot at kinematic level. Namely, u_2 is directly related to velocity $\dot{\theta}_2$, which is expressed in the inertial frame. Typically, the considered structure is equipped with an actuator responsible for moving the first link with respect to the second. Hence, the more obvious kinematic input is angular velocity $\omega := \dot{q}_2 - \dot{q}_1$. Taking advantage of (5) one can find the following globally-defined transformation

$$\omega = -(a + q_3^2)u_2 + (\eta_2 - \eta_1)\sigma, \tag{7}$$

describing the relationship between the relative angular velocity and the terms present in (5).

2.2. Lie algebra and Lie group. In this paper the concept of Lie group and Lie algebra is used extensively, [28]. This section gives selected details concerning description of control systems on Lie groups.

Firstly, we recall definition of distribution following Isidori [29]. Let Δ be a smooth distribution and p a regular point of Δ. Suppose that $\dim(\Delta(p)) = k$. Then, there exists an open neighborhood U of p and a set $\{V_1, \ldots, V_k\}$ of smooth vector fields defined on U with the property that

- the vectors $V_1(x), \ldots, V_k(x)$ are linearly independent at each x in U
- $\Delta(x) = \operatorname{span}\{V_1(x), \ldots, V_k(x)\}$ at each x in U

Moreover, every smooth vector field Z belonging to Δ can be expressed, on U, as

$$Z(x) = \sum_{i=1}^{k} c_i(x)V_i(x), \tag{8}$$

where $c_1(x), \ldots, c_k(x)$ are smooth real-valued functions of x, defined on U.

Next, we consider the Lie bracket $[Y_i, Y_j]$ of a pair of smooth vector fields. The Lie bracket satisfies antisymmetry and Jacobi identity, [30]. A *Lie algebra* of vector fields, $Lie\{Y_1, Y_2, \ldots, Y_m\}$, over $\mathbb{R}$ is a vector space spanned by generators $Y_1, \ldots, Y_m$ and all Lie brackets derived from the generators and their descendants. Let $\mathrm{Dim}_{\mathbb{R}} Lie\{Y_1, Y_2, \ldots, Y_m\}$ denote dimension of the Lie algebra determined over $\mathbb{R}$. We can say that $\mathrm{Dim}_{\mathbb{R}} Lie = k$, if k denotes maximum number of independent vector fields, $V_1, V_2, \ldots, V_k \in Lie$, belonging to this Lie algebra. These vector fields must satisfy the following implication

$$\forall x \in U, \ \sum_{i=1}^{k} c_i V_i(x) = 0 \Rightarrow c_1 = c_2 = \ldots = c_k = 0, \tag{9}$$

where c_i's are constant scalars (cf. (8) where smooth functions are taken into account). It is worth emphasizing that number of independent vector fields can be the same or higher than dimension of distribution spanned by these vector fields. Following this statement, the dimension of Lie algebra can be higher (even infinite) than dimension of space in which vector fields live.

Remark 1. To illustrate dependence of vector fields consider the following simple example. Let

$$V_1(x) = \begin{bmatrix} \cos x_1 \\ x_2 \cos x_1 \end{bmatrix}, \qquad V_2(x) = \begin{bmatrix} 1 \\ x_2 \end{bmatrix}, \tag{10}$$

be vector fields in $\mathbb{R}^2$, while $x = [x_1 \ x_2]^\top \in \mathbb{R}^2$ denotes some point on U. Then easy calculation shows that $V_1(x) = \cos(x_1)V_2(x)$, which indicates that vector fields V_1 and V_2 are locally linearly dependent $\forall x \in U$. In spite of that they are independent vector fields over $\mathbb{R}$, since there is no coefficient c (constant) such that $V_1 = cV_2$.

Now, assume that $V \in Lie\{Y_1, Y_2, \ldots, Y_m\}$ is a vector field obtained as a (repeated) Lie bracket of generators $Y_1, \ldots, Y_m$, and denote by $b_i(V)$ the number of occurrences of Y_i in V. Then we define *degree of vector field* V as $\deg(V) := \sum_{i=1}^{m} b_i(V)$. A set of independent vector fields[1] of b^{th} degree constitutes *layer* $L_b := \{V \in Lie : \deg(V) = b\}$.

Remark 2. To illustrate this point let us consider Lie algebra $Lie = \{Y_1, Y_2\}$. In such a case the following layers can be distinguished:

- layer L_1: Y_1 and Y_2
- layer L_2: $[Y_1, Y_2]$
- layer L_3: $[Y_1, [Y_1, Y_2]]$, $[Y_2, [Y_1, Y_2]]$
- layer L_4: $[Y_1, [Y_1, [Y_1, Y_2]]]$, $[Y_2, [Y_2, [Y_1, Y_2]]]$, $[Y_2, [Y_1, [Y_1, Y_2]]]$
- etc.

Next, assume that there is some positive integer $\bar{b}$ such that for $b \geq \bar{b}$ layer L_b contains only zero vector fields, namely vector fields of order greater than $\bar{b} - 1$ vanish. Then one can say that Lie algebra is *nilpotent*.

Consider a Lie group G which carries the structure of a smooth manifold, such that the group operation $G \times G \to G$, $(g, h) \to g \circ h$, as well as the inversion $G \to G$, $g \to g^{-1}$, are differentiable maps. Assume that e is neutral element of the group G such that[2] $\forall g \in G$, $ge = eg = g$.
For any $g \in G$ the following diffeomorphisms can be defined:

- left translation: $l_g : \ G \to G, \ h \to gh$;
- right translation: $r_g : \ G \to G, \ h \to hg$; and
- conjugation: $\phi_g : \ G \to G, \ h \to l_{g^{-1}}(r_g(h)) = r_{g^{-1}}(l_g(h)) = ghg^{-1}$.

Assume that $V(h) \in T_hG$ denotes vector field in the tangent space of G determined at h. Using the defined maps one can pushforward $V(h)$ to other points on G. Hence, the corresponding differential operators can be taken into account: $dl_g(h) := \dfrac{d}{dh} l_g(h)$, $dr_g(h) := \dfrac{d}{dh} r_g(h)$ and $d\phi_g(h) := \dfrac{d}{dh} ghg^{-1}$, respectively.

For any Lie group G an associated Lie algebra $\mathfrak{g}$ can be considered. In this case, the algebra $\mathfrak{g}$ is the vector space of all left invariant[3] vector fields on G satisfying

$$dl_g(h)V_i(h) = V_i(gh). \tag{11}$$

[1] Dependent vector fields can be excluded taking into account Lie bracket antisymmetry and Jacobi identity.
[2] Typically the symbol '$\circ$' is omitted to shorten the notation – for example $g \circ h$ is commonly replaced by gh.
[3] Alternatively, right invariant vector fields can be also taken into account.

From relationship (11) it follows that pushing forward any left-invariant vector field V_i at $h \in G$ to $dl_g(h)V_i(h)$ at gh can be replaced by evaluating it directly at gh. Since each left invariant vector field on G is uniquely associated with a vector tangent to H at e, we can identify the Lie algebra $\mathfrak{g}$ of G with the tangent space to G at e, $\mathfrak{g} \approx T_eG$, [31]. This implies that $\mathfrak{g}$ is a vector space of the same dimension as the underlying group. Moreover, as is convenient, we can view the Lie algebra of a Lie group either as the space of left invariant vector fields or as the tangent space to the group at the identity element.

Define the Lie algebra $\mathfrak{g}$ basis consisting of independent vector fields $X_1, X_2, \ldots, X_n$. In the given coordinates it can be described using matrix notation by $X = [X_1\ X_2\ \ldots\ X_n] \in \mathbb{R}^{n\times n}$, where $n = \dim G$ is dimension of G. Taking advantage of this notation one can express any vector fields $V \in \mathfrak{g}$ evaluated at $g \in G$ in the Lie algebra basis as: $V = X(g)w$, where $w \in \mathbb{R}^n$.

The Lie algebra $\mathfrak{g}$ of the group G is related with this group via exponential map: $\mathfrak{g} \to G$, denoted by

$$g = \exp(X(g)w), \tag{12}$$

where $g \in G$ and $X(g)w \in \mathfrak{g}$. This map is understood as the solution of differential equation $\frac{d}{d\tau}g = X(g)w$ with $g(0) = e$ evaluated at $\tau = 1$.

The other important differential operator is the adjoint operator $Ad:\ G \times \mathfrak{g} \to \mathfrak{g}$ which is given by $Ad(g)V := d\phi_g(e)V$, where $V \in \mathfrak{g}$.

Now let us recall a few basic definitions and properties concerning homogeneity [22, 32, 33]. Assume that element g of Lie group G is defined in Euclidean space, namely $g = x = [x_1\ x_2\ \ldots x_n] \in \mathbb{R}^n$, while x_i $(i = 1, \ldots, n)$ denotes a coordinate. Then one can define the dilation $\delta^r_\epsilon :\ G \to G$, where $\epsilon > 0$ is positive parameter, $r = (r_1, r_2, \ldots, r_n)$ denote the weight vector with positive integers. To be more specific, the dilation is given by:

$$\delta^r_\epsilon(x) = [\epsilon^{r_1}x_1\ \epsilon^{r_2}x_2\ \ldots \epsilon^{r_n}x_n]^\top. \tag{13}$$

Next, assume that $V(x) = \sum_{j=1}^{n} v_j(x)\frac{\partial}{\partial x_j} \in \mathfrak{g}$ is a vector field, where $v_j(x)$ is a real function. This vector field is homogeneous with degree s with respect to dilation δ^r_ϵ if the following relationship is satisfied:

$$V\left(\delta^r_\epsilon(x)\right) = \epsilon^s \sum_{j=1}^{n} \epsilon^{r_j} v_j(x)\frac{\partial}{\partial x_j}. \tag{14}$$

Remark 3. To exemplify this definition we consider two vector fields in $\mathbb{R}^3$:

$$V_1(x) = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \left(x_2(x_1+x_2)+x_3\right)\frac{\partial}{\partial x_3}$$

and

$$V_2(x) = \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3}$$

and weight vector $r = (1, 1, 2)$. From (13) we have $\delta^r_\epsilon(x) = \left[\epsilon x_1\ \epsilon x_2\ \epsilon^2 x_3\right]^\top$. Next, evaluating both vector fields at $\delta^r_\epsilon(x)$ we have

$$V_1(\delta^r_\epsilon(x)) = \epsilon x_1\frac{\partial}{\partial x_1} + \epsilon x_2\frac{\partial}{\partial x_2} + \epsilon^2\left(x_2(x_1+x_2)+x_3\right)\frac{\partial}{\partial x_3}$$

and

$$V_2(\delta^r_\epsilon(x)) = \epsilon\frac{\partial}{\partial x_1} + \epsilon^2 x_2\frac{\partial}{\partial x_3}.$$

Then, it is straightforward to show that $V_1(\delta^r_\epsilon(x))$ and $V_2(\delta^r_\epsilon(x))$ satisfy (14) for $s = 0$ and $s = -1$, respectively. Hence, one can conclude that V_1 and V_2 are homogenous vector fields with degree 0 and -1, respectively, for the given dilation.

Next, consider m input control affine driftless system

$$\dot{x} = \sum_{i=1}^{m} X_i(x)u_i, \tag{15}$$

with x belonging to a n-dimensional smooth manifold M, while $X_1, \ldots, X_m$ being smooth *control vector fields* and $u_1, \ldots, u_m$ denoting bounded inputs. It is important to notice that the satisfaction of the group property may be local only i.e. in a neighborhood $U \subset M$ of a given point $x \in U$. In the special case when the Lie algebra is nilpotent, there is an equivalence between the local and global satisfaction of this property.

Claim 1. Assume that dimension of the control Lie algebra $k = \mathrm{Dim}_{\mathbb{R}} Lie\{X_1, \ldots, X_m\}$ of system (15). Then the following cases can be considered:

1. $k = n$ – the system (15) can be defined on a Lie group;
2. $k = \overline{n} > n$ – the system (15) cannot be defined on a Lie group explicitly but it is possible to do extension of the state space such that the extended system is defined on a Lie group;
3. $k = \infty$ – the system (15) cannot be defined globally on any Lie group.

Next, we consider the first case for which dimension of the control Lie algebra is finite and equals n. Then it is possible to find a Lie group – for example using algorithm outlined in [34]. Motivated by this idea we introduce the following lemma.

Lemma 1 (Determination of a Lie group structure for the particular control Lie algebra). Let g and $h \in \mathbb{R}^n$ be defined in n-dimensional Euclidean space. Simultaneously, it is assumed that g and h are elements of some Lie group G with unknown operation $'\circ'$ and neutral element $e = 0 \in \mathbb{R}^n$. Next, consider a n-dimensional nilpotent Lie algebra $\mathfrak{g}$ of group G. Let $X \in \mathbb{R}^{n\times n}$ denote basis of this algebra in the matrix form, which is composed of n linearly left-invariant independent vector fields. Then the group operation $'\circ'$ can be found by solving the following integral equation:

$$g \circ h = g + \int_0^1 X\left(g \circ (sh)\right) X^{-1}(sh)\, h ds, \tag{16}$$

where $s \in \mathbb{R}$.

Proof 1. Since vector fields in $X \in \mathbb{R}^{n\times n}$ are left-invariant the following relationship is satisfied

$$X\left(g\circ h\right) = dl_g(h)X(h). \tag{17}$$

Next, introduce an independent variable $s \in [0, 1]$ and consider the following left translation

$$g\circ sh = l_g(sh). \tag{18}$$

Taking derivative of Eq. (18) with respect to variable s one has: $\frac{d}{ds}l_g(sh) = dl_g(sh)h$ or equivalently

$$dl_g(sh) = dl_g(sh)hds. \tag{19}$$

Then, integrating both sides of (19) one has

$$l_g(h) - l_g(0) = \int_0^1 dl_g(sh)hds. \tag{20}$$

Since, from definition $l_g(0) = g$, one can write that

$$l_g(h) = g + \int_0^1 dl_g(sh)hds. \tag{21}$$

From (17) one has $dl_g(h) = X\left(l_g(h)\right)X^{-1}(h)$. Using this result in (19) and noticing that h is substituted by sh we obtain:

$$gh = g + \int_0^1 X\left(g\circ (sh)\right)X^{-1}(sh)hds. \tag{22}$$

From (22) the group operation $'\circ'$ can be calculated.

2.3. Dynamic extension and description of system kinematics on a Lie group. It is assumed that the control law discussed in this paper should be applied to kinematics (5) with zero or non-zero angular momentum σ. Basically, it means that the drift term in (5) is not necessary for proper operation of the controller. Hence, the design methodology of the control algorithm is based on vector fields associated to control inputs u_1 and u_2. Simultaneously, the drift in (5) can be considered as a disturbance which can be partly compensated by the controller taking advantage of a feed-forward loop.

Following this assumption, we deal with system (5) for the driftless case when $\sigma = 0$. Consequently, the kinematics is simplified as follows

$$\dot{q} = \overline{X}_1 u_1 + \overline{X}_2(q)u_2, \tag{23}$$

where $\overline{X}_1 := \frac{\partial}{\partial q_3}$ and $\overline{X}_2 := q_3^2\frac{\partial}{\partial q_1} - a\frac{\partial}{\partial q_2}$ denote vector fields generators. To investigate properties of system (23) we refer to its Lie algebra $Lie\left\{\overline{X}_1, \overline{X}_2\right\}$ generated by $\overline{X}_1$, $\overline{X}_2$ and the corresponding Lie brackets. Noticing that coordinates of vector fields $\overline{X}_1$ and $\overline{X}_2$ are given by polynomials functions it can be concluded that the Lie algebra is nilpotent. Indeed, considering layers of the Lie algebra (cf. Subsec. 2.2) it can be found that $\left[\overline{X}_2, \left[\overline{X}_1, \overline{X}_2\right]\right] = 0$ and $\forall V \in Lie\left\{\overline{X}_1, \overline{X}_2\right\}$ such that $\deg(V) \geq 4$, $V \equiv 0$. Correspondingly, non-zero vector fields in this algebra span the following distribution:

$$\overline{\Delta}(q) = \text{span}\left\{\overline{X}_1, \overline{X}_2, \left[\overline{X}_1, \overline{X}_2\right], \left[\overline{X}_1, \left[\overline{X}_1, \overline{X}_2\right]\right]\right\}.$$

It can be shown that the dimension of the distribution of $\forall q \in \mathcal{Q}, \dim\overline{\Delta}(q) = \dim\mathcal{Q} = 3$, which implies that the driftless system (23) is small time locally controllable.

Referring to conditions of existence of a Lie group for the given control system, outlined in Subsec. 2.2, we can conclude that in the considered case they are satisfied. Namely, Lie algebra $Lie\left\{\overline{X}_1, \overline{X}_2\right\}$ is nilpotent that implies a finite exponential representation of vector fields (cf. (12)), defining a manifold M. This, in turn, allows finding some symmetry and a Lie group G, on which the dynamics of the system (23) can be described.

However, in the present case this issue is more complex since assumptions for Lemma 1 are not satisfied. One can notice that four vector fields in distribution $\overline{\Delta}(q)$ are linearly independent over $\mathbb{R}$ in spite of the fact, that $\forall q \in \mathcal{Q}, \dim\overline{\Delta}(q) = 3$. Hence, the dimension of Lie algebra $Lie\left\{\overline{X}_1, \overline{X}_2\right\}$ is $r = 4$. Then assuming that G is a Lie group with the given Lie algebra Lie, the dimension of G would also be r. In such a case exponential map (12) would be a diffeomorphism. However, when Lie algebra basis is constituted by more independent vector fields than the dimension of space in which these vector fields live (notice that $\overline{X} \in \mathbb{R}^n$, where $n < r$), exponential map becomes surjection and the group G cannot be defined globally. In this case we have to consider the case 2 outlined in Claim 1.

This issue can be solved referring to Rothschild-Stein lifting theorem [35]. Noticing that $n < r$ one can propose a dynamic extension of system (23). In order to do this, we define a new configuration $g = [g_1\ g_2\ g_3\ g_4]^\top \in \mathcal{Q}\times\mathbb{R}$, where

$$g_1 := q_3,\ g_2 := q_2,\ g_4 := q_1, \tag{24}$$

while g_3 is an additional variable that does not correspond to any original coordinate in $q \in \mathcal{Q}$. The selection of the additional dynamics governing evolution of g_3 is not unique. Motivated by similarity of kinematics (23) to a chained system (cf. [9]) one can propose the following relationship

$$\dot{g}_3 = g_1 u_2. \tag{25}$$

Next, complementing the system (23) with auxiliary dynamics (25) one has

$$\dot{g} = X_1 u_1 + X_2(g)u_2, \tag{26}$$

where

$$X_1 := \frac{\partial}{\partial g_1}, \qquad X_2 := -a\frac{\partial}{\partial g_2} + g_1\frac{\partial}{\partial g_3} + g_1^2\frac{\partial}{\partial g_4}$$

are fundamental vector fields. As in the previous case, we consider non-trivial vector fields of nilpotent Lie algebra $Lie\left\{X_1, X_2\right\}$. These vector fields span the following distribution: $\Delta(q) = \text{span}\left\{X_1, X_2, X_3, X_4\right\}$, where $X_3 := [X_1, X_2]$ and $X_4 := [X_1, [X_1, X_2]]$. Then, $\forall q \in \mathcal{Q}, g_3 \in \mathbb{R}, \dim\Delta(g) = 4$, namely the dimension of the distribution Δ is the same as the dimension of the algebra $Lie\left\{X_1, X_2\right\}$. Hence, the vector fields in distribution Δ can be seen as the basis of the Lie algebra $Lie\{X_1, X_2\}$. Using the matrix notation and calculating Lie brackets we define this basis as follows

$$X(g) := [X_1\ X_2\ X_3\ X_4](g) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & g_1 & 1 & 0 \\ 0 & g_1^2 & 2g_1 & 2 \end{bmatrix} \in \mathbb{R}^{4\times 4}. \tag{27}$$

Consequently, one can find a group G such that $\forall g \in G$, $\dim G = 4$ referring to Lemma 1. After some calculations (the details of which can be found in Appendix) the following group operation can be specified

$$gh = \begin{bmatrix} g_1 + h_1 \\ g_2 + h_2 \\ g_3 + h_3 - \frac{1}{a} g_1 h_2 \\ g_4 + h_4 - \frac{1}{a} g_1^2 h_2 + 2 g_1 h_3 \end{bmatrix}, \tag{28}$$

where $g, h \in G$. From now one can say that system (26) is defined on Lie group G. Obviously, this is the system with a deficit of control – assuming that $m = \dim u$ for the considered case one has $n - m = 2$.

Another important property of system (26) can be found, namely it can be proved that its fundamental vector fields are homogeneous of degree $p = -1$. Consequently, one can consider a dilation δ_ϵ^r defined on any $g \in G$, where $r = (1, 1, 2, 3)$ determines the weight vector.

Finally, we return to a more general case concerning description of the robot kinematics with non-zero angular momentum σ. Recalling the original kinematics with drift (5) and the new set of coordinates given by (24), kinematics (26) can be rewritten as follows

$$\dot g = X_0 + X_1 u_1 + X_2(g) u_2, \tag{29}$$

where

$$X_0 := \sigma\eta_2 \frac{\partial}{\partial g_2} + \sigma\eta_1 \frac{\partial}{\partial g_4}. \tag{30}$$

Alternatively, vector fields in (29) can be expressed in the Lie algebra basis $X(g)$. Then the considered kinematics takes the following form

$$\dot g = X(g)\,(Cu + \nu)\,, \tag{31}$$

where

$$C := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^\top \in \mathbb{R}^{4\times 2} \tag{32}$$

and $\nu := X(g)^{-1} X_0$.

3. Control algorithm

Let us formally define the control problem.

Problem 1 (Control problem). Let $g_r(t) \in G$ define desired bounded configuration and satisfy the following differential equation

$$\dot g_r = X(g_r) v_r, \tag{33}$$

where X is defined by (27) and $v_r \in \mathbb{R}^4$ denotes a bounded reference input.

Find bounded input u of kinematics (31), such that for any bounded angular momentum σ configuration error becomes bounded as: $\lim_{t\to\infty} \|g(t) - g_r(t)\| \le \epsilon$, where $\epsilon > 0$ is an arbitrarily small radius of neighborhood of zero.

3.1. Transverse function. The control method proposed in this paper is based on transverse functions whose derivative virtually adds new directions in the tangent space not covered directly by the fundamental vector fields of the control process. It gives possibility to recover approximately motion in directions directly inaccessible due to nonholonomic constraints.

In this section we give a brief description of the synthesis of transverse functions for the system (26). More details and examples can be found in [9, 12, 32].

Consider that the small-time controllable system (26) is described on Lie group G and the deficit of control inputs equals $n - m$. A nominal transverse function $f(\alpha)$ can be defined on the torus $\mathbb{T}^{n-m}$, namely $\alpha = [\alpha_1\ \alpha_2]^\top \in \mathbb{T}^2$. Taking advantage of the group operation the transverse function can be calculated as follows

$$f(\alpha) := f_{II}(\alpha_2) f_I(\alpha_1), \tag{34}$$

where

$$f_I(\alpha_1) = \exp\left(X_1 \beta_{1,1} \sin\alpha_1 + X_2 \beta_{1,2} \cos\alpha_1\right),$$
$$f_{II}(\alpha_1) = \exp\left(X_1 \beta_{2,1} \sin\alpha_2 + X_3 \beta_{2,2} \cos\alpha_2\right)$$

denotes exponential maps of the selected vector fields, $\beta_{i,j} \in \mathbb{R}$, $i, j = 1, 2$ are parameters – cf. [32, 36]. Basically, the components f_I and f_{II} are related to directions determined by higher order Lie brackets X_3 and X_4, respectively. Making the detailed calculations one can obtain the following result:

$$f(\alpha) = \begin{bmatrix} \beta_{1,1}\sin\alpha_1 + \beta_{2,1}\sin\alpha_2 \\ -a\beta_{1,2}\cos\alpha_1 \\ \frac{1}{4}\beta_{1,1}\beta_{1,2}\sin 2\alpha_1 + \beta_{2,2}\cos\alpha_2 + \beta_{2,1}\sin\alpha_2\cos\alpha_1 \\ f_4 \end{bmatrix}, \tag{35}$$

where

$$f_4 := \frac{1}{6}\beta_{1,1}^2\beta_{1,2}\sin\alpha_1\sin 2\alpha_1 + \frac{1}{2}\beta_{2,1}\beta_{2,2}\sin 2\alpha_2 + \beta_{2,1}^2\beta_{1,2}\sin^2\alpha_2\cos\alpha_1 + \frac{1}{2}\beta_{1,1}\beta_{1,2}\beta_{2,1}\sin\alpha_1\sin 2\alpha_1. \tag{36}$$

The parameters $\beta_{i,j}$ should be selected in order to satisfy the transversality condition, which in the considered case can be represented as

$$\forall\alpha \in \mathbb{T}^2, \det A_2(f(\alpha))) \ne 0, \tag{37}$$

where

$$A(f) := \left[A_1^\top\ A_2^\top\right]^\top (f) = X^{-1}(f) \frac{\partial f}{\partial \alpha} \in \mathbb{R}^{4\times 2}, \tag{38}$$

while $A_1, A_2 \in \mathbb{R}^{2\times 2}$.

In order to simplify selection of parameters $\beta_{i,j}$ we consider a scaled transverse function f_ϵ using dilation operator δ_ϵ^r:

$$f_\epsilon := \delta_\epsilon^r(f). \tag{39}$$

Thanks to the homogeneity of the vector fields (23) the dilation preserves transversality of function f [35,36]. To be more specific, if f is a function of transverse, then the property is transferred to the function f_ϵ. Additionally, one can assume that parameter ϵ in dilation $\delta_\epsilon^r(f(\alpha))$ is time varying. Then time derivative of f_ϵ can be presented as follows

$$\dot{f}_\epsilon = X(f_\epsilon)\left(A(f_\epsilon)\dot{\alpha} + A_\epsilon(f_\epsilon)\dot{\epsilon}\right), \tag{40}$$

where $A_\epsilon(f_\epsilon) := X^{-1}(f)\frac{\partial f_\epsilon}{\partial \epsilon} \in \mathbb{R}^4$.

3.2. Control law synthesis. In order to design the controller we use the symmetry of the control system (31). Consequently, the configuration error is defined on Lie group G as follows: $\widetilde{g} := g_r^{-1}g$. Taking time derivative of $\widetilde{g}$, using (31), the reference model (33), and referring to calculations presented in [12] one can obtain the following error dynamics

$$\dot{\widetilde{g}} = X(\widetilde{g})\left(C(g)u + \nu - Ad^X\left(\widetilde{g}^{-1}\right)\nu_r\right), \tag{41}$$

where $Ad^X(g) := X(e)^{-1}Ad(g)X(e)$ is the adjoint operator expressed in the Lie algebra basis X.

Next, to quantify a tracking of transverse function an auxiliary error is introduced: $z := \widetilde{g}f_\epsilon^{-1}$. This error is governed by the following dynamics (cf. [11,12]):

$$\dot{z} = X(z)Ad^X(f_\epsilon)\left(\overline{C}\overline{u} + \nu - Ad^X\left(\widetilde{g}^{-1}\right)\nu_r - A_\epsilon\dot{\epsilon}\right), \tag{42}$$

where $\overline{C} := [C \;\; -A(f_\epsilon)] \in \mathbb{R}^{4\times4}$ and $\overline{u} := \left[u^\top \; \dot{\alpha}^\top\right]^\top \in \mathbb{R}^4$ is an extended input consisting of real input u and the auxiliary input $\dot{\alpha}$ governing time evolution of transverse functions f_ϵ.

To stabilize the open-loop system (42) we refer to the results presented in [12, 36]. Basically, we use a suboptimal controller which makes it possible to optimize control effort based on designed quadratic term. In this case, we consider the optimization of control, taking the u_1 and ω into account. This is motivated by practical meaning of ω, which has been discussed in Subsec. 2.1.

Proposition 1 (Suboptimal controller). Let $J(\overline{v}) = \frac{1}{2}\overline{v}^\top W_2\overline{v}$, be the performance measure, where $\overline{v} := \left[u_1 \; \omega \; \dot{\alpha}^\top\right]^\top \in \mathbb{R}^4$ is the extended input (cf. definition of $\overline{u}$), $W_2 \in \mathbb{R}^{4\times4}$ is a positive definite matrix. Let us define

$$H := X(z)Ad^X(f_\epsilon), \qquad Q := H\overline{C}T, \tag{43}$$

where $T := \operatorname{diag}\left\{1 - \frac{1}{a+g_1^2}, 1, 1\right\} \in \mathbb{R}^{4\times4}$ and assume that $\nu_s^c := -Q^{-1}W_1 z$, $\nu_d^c := -T^{-1}\overline{C}^{-1}p$, $\nu_s^o := -z^\top W_1 z W_2{}^{-1}Q^\top z$ and $\nu_d^o := -z^\top H p W_2{}^{-1}Q^\top z$, with $p := \nu - Ad^X\left(\widetilde{g}^{-1}\right)\nu_r - A_\epsilon\dot{\epsilon}$, where the indices 's' and 'd' describe the static and dynamic terms, while $W_1 \in \mathbb{R}^{4\times4}$ is a positively defined gain matrix. The control algorithm defined as follows

$$\overline{v} = \frac{\lambda_s}{y+\lambda_s}\nu_s^c + \frac{1}{y+\lambda_s}\nu_s^o + \frac{\lambda_d}{y+\lambda_d}\nu_d^c + \frac{1}{y+\lambda_d}\nu_d^o, \tag{44}$$

where λ_s and $\lambda_d > 0$ are positive coefficients, and $y := z^\top Q W_2{}^{-1}Q^\top z$, applied to (29) considering the transformation input (7), provides exponential stability of the equilibrium point $z = e$ and optimizes instantaneous control input, thus minimizing of the functional $J(\overline{v})$ when λ_s and $\lambda_d \to 0$.

Remark 4. It can be noted that when λ_s and $\lambda_d \to \infty$ the algorithm given by (44) becomes a classic controller based on decoupling of nonlinear system (42). In such a case the closed-loop system is given by $\dot{z} = -W_1 z$ and the optimization problem is not taken into account. In contrast, when λ_s and $\lambda_d \to 0$ the optimization goal is considered more thoroughly but the controller is more sensitive to unmodeled dynamics, [36]. Hence, one can select coefficients λ_s and λ_d in order to adjust optimization level and the controller robustness.

Remark 5. The controller (44) is defined globally for the extended system (31). However, this property is not fully transferred to kinematics (5). Recalling configuration space of this kinematics one has $d > 0$, which is well motivated geometrically and physically, namely distance d should be positive and minimum inertia of the second of the robot cannot be reduced to zero. However, this configuration constraint is not ensured by the proposed controller. This constraint can be violated especially during transient stage when auxiliary error z is significant. In order to cope with this issue ϵ should be made relatively small and initial configuration error should be reduced. It can be quite easily met by designing an appropriate reference trajectory such that $g_r(0)$ is close enough to $g(0)$.

4. Simulation results

In order to verify properties of the algorithm, numerical simulations were performed in *Matlab/Simulink* environment. The parameters of the kinematics (5) were selected as follows: $I_1 = 1\,\mathrm{kg}\cdot\mathrm{m}^2$, $a = 0.5\,\mathrm{m}^2$ with the drift model for which $\eta_1 = \frac{1}{I_1}$ and $\eta_2 = 0$. The nominal parameters of the controller were chosen as: $\epsilon = 0.4$, $\beta_{1,1} = 0.67$, $\beta_{1,2} = 0.45$, $\beta_{2,1} = 1.33$, $\beta_{2,2} = 0.2$, $\alpha(0) = 0$, $W_1 = \operatorname{diag}\{4,4,2,2\} \in \mathbb{R}^{4\times4}$, $W_2 = \operatorname{diag}\left\{0.1, 1, 10^{-5}, 10^{-6}\right\} \in \mathbb{R}^{4\times4}$. Simulations were performed for the following initial conditions: $\theta_1(0) = \frac{\pi}{2}$ rad, $\theta_2(0) = 0$ rad, $d(0) = 1$ m, $g_3(0) = 0$, $\theta_{r1} = 0$ rad, $\theta_{r2} = \frac{\pi}{2}$ rad, $d_r = 2$ m and $g_{r3} = 0$.

The following simulation scenarios were considered:

- S1: Stabilization without optimization for zero angular momentum $\sigma = 0$: λ_s and $\lambda_d \to \infty$.
- S2: Stabilization without optimization for zero angular momentum $\sigma = 0$ assuming scaling of transverse function: λ_s and $\lambda_d \to \infty$, $\epsilon(t) = e^{-0.5t} + 0.3$.
- S3: Stabilization with optimization for zero angular momentum $\sigma = 0$: $\lambda_s = \lambda_d = 1$.
- S4: Stabilization with increased gains to $W_1 = \operatorname{diag}\{20, 20, 10, 10\}$, other conditions as in S3.

- S5: Stabilization with optimization for non-zero angular momentum $\sigma = 0.2\,\mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}$ for $t \in [0,5)$s and $\sigma = -0.2\,\mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}$ for $t \in (5,10]$s, other conditions as in S3.
- S6: Tracking of non-feasible reference trajectory for zero angular momentum $\sigma = 0$: $\dot{\theta}_{r1}(t) = \dot{\theta}_{r2}(t) = 0.5\cos(0.2\pi t)\,\mathrm{rad/s}$, $\dot{d} = \dot{g}_{r3} = 0$, other conditions as in S3.
- S7: Tracking of non-feasible reference trajectory for non-zero angular momentum $\sigma = 0.2\,\mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}$: other conditions as in S6.

Considering the results of simulation S1 presented in Fig. 2 one can see that for the basic version of the controller, when no optimization is used, the transient stage is highly oscillatory. The configuration error tends to some neighborhood of zero that can be adjusted by parameter $\epsilon > 0$. In the considered case the assumed radius of the neighborhood is quite large in order to limit magnitude of control inputs. Comparing Figs. 2b and 2c one can observe that error $\widetilde{g}$ defined on Lie group is scaled significantly. In particular value of $\widetilde{g}_4$ is increased (notice that coordinate $g_4 = \theta_1$). Since changing of coordinate g_4 requires generation of higher order Lie bracket, highly oscillatory signals observed in the considered case can be easily understood.

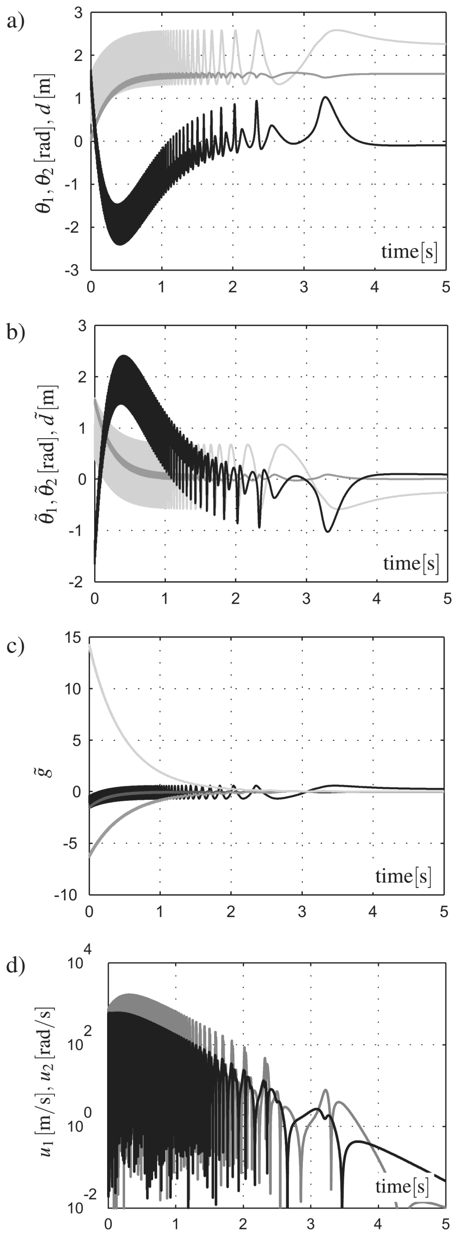

Fig. 2. Simulation S1: a) configuration: θ_1 (■), θ_2 (■), d (■), b) configuration error $q - q_r$: $\widetilde{\theta}_1 := \theta_1 - \theta_{r1}$ (■), $\widetilde{\theta}_2 := \theta_2 - \theta_{r2}$ (■), $\widetilde{d} := d - d_r$ (■), c) configuration error $\widetilde{g}$: $\widetilde{g}_1$ (■), $\widetilde{g}_2$ (■), $\widetilde{g}_3$ (■), $\widetilde{g}_4$ (■), d) control inputs: u_1 (■), ω (■)

In Fig. 3 the results of stabilization with time-varying ϵ are presented. In this case oscillatory behavior is reduced by using higher initial value of ϵ which enables limiting control effort considerably. The disadvantage of this approach can be observed as an rapid increase of some configuration variables during regulation process. This is due to the fact that for higher ϵ the auxiliary trajectory given by the transverse function is contained in the neighborhood of zero with a bigger radius. As it can be noticed from Fig. 3a variable d becomes negative in some period of time. Consequently, the coordinate constraint is violated.

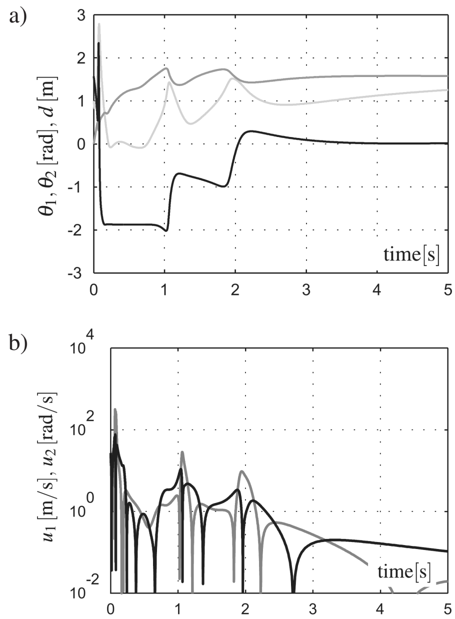

Fig. 3. Simulation S2: a) configuration, b) control inputs (detailed description is given below Fig. 2)

The next simulation (cf. Fig. 4), S3, was obtained assuming optimization of control effort. As in the previous case, the oscillatory time response of the closed loop system is reduced. In particular, one can notice a significant change of coordinate θ_1. In comparison to simulation S2 one can see that d does not go to zero at any time instant.

The regulation time can be adjusted by increasing the controller gains (specified by W_1). It can be observed from Fig. 5 that by increasing gains coefficients five times the duration of transient stage has been reduced respectively. It turns out that the time plots of configuration variables are scaled in time with respect to results obtained in S3. Similarly, instantaneous values of inputs u_1 and ω are increased.

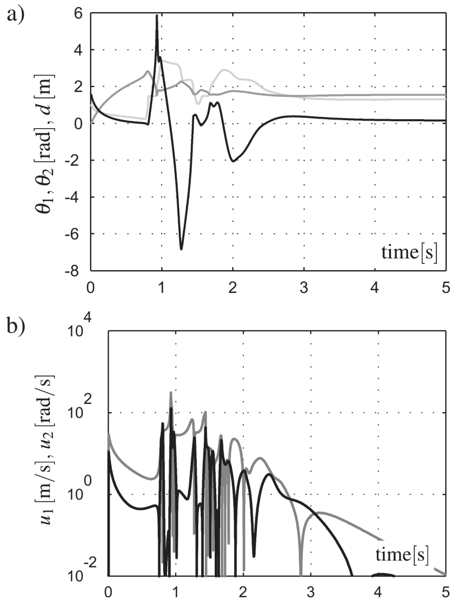

Fig. 4. Simulation S3: a) configuration, b) control inputs (detailed description is given below Fig. 2)

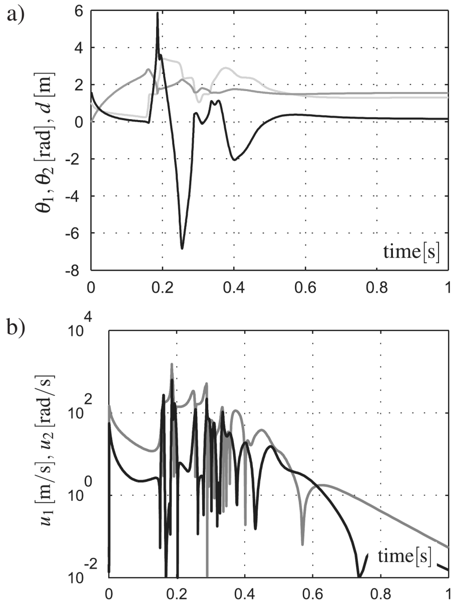

Fig. 5. Simulation S4: a) configuration, b) control inputs (detailed description is given below Fig. 2)

In order to estimate control effort the following integral is considered: $\int\limits_0^5 \| [u_1 \ \omega]^\top (t) \|^2 dt$. Values of this integral calculated for simulation S1, S2 and S3 are: $2.04 \cdot 10^5$, $1.5 \cdot 10^3$ and $2.3 \cdot 10^3$, respectively. It confirms that for the similar regulation time the control magnitude can be decreased by using tuning of parameter ϵ and taking advantage of gains optimization. For simulation S4, when duration of transient stage is reduced, the integral index achieves $1.16 \cdot 10^4$. It means that energetic effort is still less than for simulation S1 with longer regulation time but without optimization.

In simulation S5 the stabilization in the presence of the permanent non-zero drift was investigated. For comparison a step change of the angular momentum was applied at 5th s. From Fig. 6 it can be concluded that the configuration errors are bounded in spite of the constant drift (for positive or negative value of σ). Magnitude of this error can be made lower, however, it would imply higher control effort. From Fig. 6b it can be seen that frequency of control signals is relatively high and does not converge to zero (the disturbance is non vanishing). This phenomenon is necessary to maintain the coordinate variables in the assumed range.

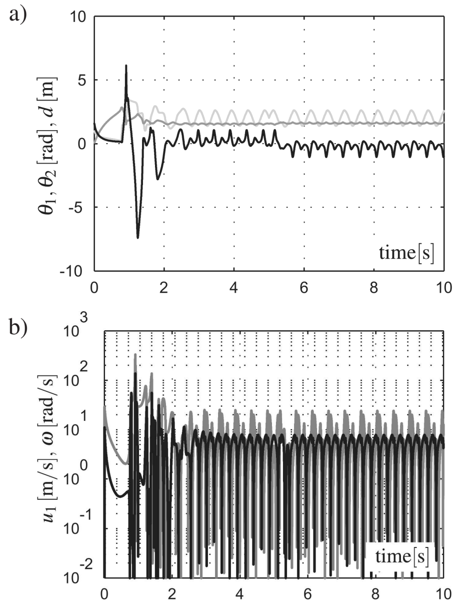

Fig. 6. Simulation S5: a) configuration, b) control inputs (detailed description is given below Fig. 2)

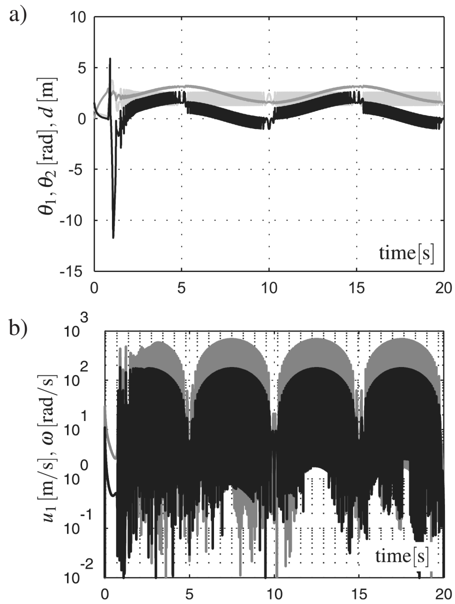

Fig. 7. Simulation S6: a) configuration, b) control inputs (detailed description is given below Fig. 2)

Simulations S6 and S7 were performed assuming tracking of a non-feasible trajectory for zero or non-zero angular momentum, respectively. It was assumed that reference motion describe change of orientation of link 1 and 2 in the same manner. This kind of motion cannot be executed directly. As a result the highly oscillatory input signals are generated. From Fig. 7 and 8 it can be inferred that the considered control task is properly realized independent on value of σ, namely the reference trajectory is approximated with the given accuracy.

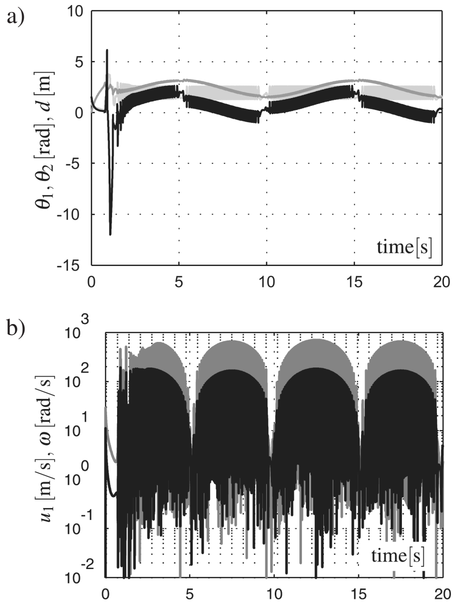

Fig. 8. Simulation S7: a) configuration, b) control inputs (detailed description is given below Fig. 1)

5. Summary

This paper presents how a relatively new control methodology can be addressed for stabilization of nonholonomic kinematics with a permanent drift. It is shown that transverse functions can be applied to such systems. The advantage of the given algorithm is that it is possible to approximate of movements which are non admissible directly. As a result the system can be stabilized around any bounded reference trajectory. In the same way any bounded drift (even an unmatched disturbance) can be compensated approximately.

It is worth emphasizing that this control scheme typically leads to highly oscillatory signals in the closed-loop system. The presented simulations examples indicate that stabilization of the considered kinematics around non-feasible reference trajectory with non-zero momentum is a very difficult control task. It is evident that when high tracking precision is assumed an appearance of oscillatory behavior cannot be easily avoided.

Other issues are related to practical aspects of control goals considered in this paper. In fact they are non-trivial and in some cases cannot be achieved as a result of limitation introduced by a real physical robot. In particular highly oscillatory controls can be very difficult to apply because of torque and force saturation. Then one can consider relaxation of the desired tracking precision to guarantee proper realization of the control task.

Still, important disadvantage of the proposed controller is difficulty of its tuning in order to improve dynamic properties during transient stage. Basically, one can manipulate gain values as well as parameters of the transverse functions. However, it is hard to establish general and precise rules how to guarantee the best control performance. This is a result of complexity of the control law and many parameters available. Moreover, the nonlinear scaling of errors on Lie group as well as the transverse function make difficult to cope with the time evolution of selected coordinates in the original space. Based on simulations, it appears that the proposed algorithm can be used rather as a local controller, when the initial configuration error is not significant. Then one can takes advantage of stability ensured by this control law, which is guaranteed even for a bounded persistence disturbance.

We believe that the presented control method can be useful for some classes of mechanical systems encountered in robotics.

The future work can be concentrated on possibility of asymptotic stabilization at least for the selected cases when the perturbation is vanishing. In such a case one can combine a trajectory planner with a closed-loop controller in order to obtain acceptable control performance.

Appendix

Derivation of group operation. Here we take advantage of Lemma 1 For control system (26) with Lie algebra basis defined by (27) one has

$$X\left(g\circ(sh)\right)X^{-1}(sh)=$$
$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -\frac{1}{a}\left((g\circ(sh)-sh_1)_1\right) & 1 & 0\\ 0 & -\frac{1}{a}\left((g\circ(sh))_1-sh_1\right)^2 & 2\left((g\circ(sh))_1-sh_1\right) & 1\end{bmatrix}, \tag{45}$$

where $(\xi)_i$ denotes i^{th} coordinate of vector ξ. From (16) and (45) one can easily find that $(g\circ(sh))_1 = g_1 + sh_1$ and $(g\circ(sh))_2 = g_2 + sh_2$. Then, coordinates $(g\circ(sg))_3$ and $(g\circ(sh))_4$ can be directly calculated by solving the following integrals:

$$\begin{aligned}(g\circ(sh))_3 &= g_3+\int_0^1\left(-\frac{1}{a}\left(g_1+sh_1-sh_1\right)h_2+h_3\right)ds\\ &= g_3+h_3-\frac{1}{a}g_1h_2,\end{aligned} \tag{46}$$

and

$$\begin{aligned}(g\circ(sh))_4 \;&=\; g_4+\int_0^1\Big(-\frac{1}{a}\left((g_1+sh_1)-sh_1\right)^2 h_2\\ &\quad +2\left(g_1+sh_1-sh_1\right)h_3+h_4\Big)\,ds\end{aligned} \tag{47}$$

$$= g_4+h_4-\tfrac{1}{a}g_1^2h_2+2g_1h_3. \tag{48}$$

Finally, relationship (28) can be obtained.

Selected operators defined for the control system. Inverse element of $g := [g_1\ g_2\ g_3\ g_4]^\top$:

$$g^{-1} = \begin{bmatrix} -g_1 \\ -g_2 \\ -g_3 - \dfrac{1}{a} g_1 g_2 \\ -g_4 + \dfrac{1}{a} g_1^2 g_2 + 2 g_1 g_3 \end{bmatrix}. \tag{49}$$

Adjoint operator:

$$Ad(g) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \dfrac{1}{a} g_2 & -\dfrac{1}{a} g_1 & 1 & 0 \\ -2g_3 & -\dfrac{1}{a} g_1^2 & 2g_1 & 1 \end{bmatrix}. \tag{50}$$

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