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On reconstructing unknown characteristics of a nonlinear system of differential equations

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Problems of dynamical reconstruction of unknown characteristics for nonlinear equations described the process of diffusion of innovations through results of observations of phase states are considered. Solving algorithms, which are stable with respect to informational noises and computational errors, are designed. The algorithms are based on the principle of auxiliary models with adaptive controls.

Key words: nonlinear differential equations, dynamical reconstruction.

1. Introduction. Statement of the problems

The problems of dynamical reconstruction on the basis of available information on an object are well known in engineering and scientific research. Our goal is to describe some algorithms for reconstructing unknown varying characteristics of dynamical systems described by second order differential equations. These algorithms are expected to be used in real time, in other words, they must be dynamical. The information on initial data is uncertain and, in general, time-varying. The algorithms should be regularizing in the sense that the final result improves if the input information becomes more accurate.

It should be noted that system (1) considered below was introduced in [1, 2] for describing the process of diffusion of innovations in a social medium.

So, we consider a system described by the equations:

$$\begin{aligned} \dot{x}_1(t) &= k(t)x_2(t) + x_1(t)(\lambda x_2(t) - \nu), \\ \dot{x}_2(t) &= -k(t)x_2(t) - (\lambda x_1(t) + \mu)x_2(t) + \gamma(t), \end{aligned} \tag{1}$$

$$t \in T = [t_0, \vartheta], \quad x_1(t_0) = x_{10}, \quad x_2(t_0) = x_{20}.$$

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Assume that positive constants λ, ν, μ, and a non-negative function $k(\cdot)$ are known whereas but the function $x_2(\cdot)$ or the function $\gamma(\cdot)$ are uncertain. We consider the situation when function $\gamma(t)$ (Lebesque measurable and satisfying the condition $\gamma(t) \in P = [-\gamma, \gamma]$, $t \in T$) acts upon the system. Here, $\gamma = \text{const} \in (0, +\infty)$. At discrete time moments

$$\tau_i \in \Delta = \{\tau_i\}_{i=0}^{m}, \quad \tau_{i+1} = \tau_i + \delta, \quad \tau_0 = t_0, \quad \tau_m = \vartheta,$$

some value $z(\tau_i) \in \mathbb{R}^n$, $n = 1$ or 2, is inaccurately measured. Results of measurements (elements $\xi_i^h \in \mathbb{R}^n$) satisfy the inequalities

$$|z(\tau_i) - \xi_i^h|_n \leqslant h, \tag{2}$$

where $h \in (0,1)$ is a level of informational noise, $|x|_1 = |x|$ is the modul of the number x, $|y|_2 = \max\{|y_1|, |y_2|\}$ is the norm of the vector $y = \{y_1, y_2\} \in \mathbb{R}^2$. We consider two cases. In the first case, we assume that the coordinate $x_1(\tau_i)$ is measured at the moment τ_i, i. e.,

$$z(\tau_i) = x_1(\tau_i), \quad \xi_i^h \in \mathbb{R}. \tag{3}$$

In the second one, the pair of coordinates $x_1(\tau_i)$ and $x_2(\tau_i)$ are measured. Then,

$$z(\tau_i) = \{x_1(\tau_i), x_2(\tau_i)\}, \quad \xi_i^h = \{\xi_{1i}^h, \xi_{1i}^h\} \in \mathbb{R}^2. \tag{4}$$

The problems under consideration consist in the following. It is required to design an algorithm allowing us to reconstruct the unknown coordinate $x_2(\cdot)$ (Problem 1) or (the second case) the input $\gamma(\cdot)$ (Problem 2). This is the meaningful statement of problems being investigated in the present paper.

Hereinafter, we assume that the following condition is fulfilled.

Condition 1 a) *The real input* $\gamma = \gamma(t)$ *generates the solution* $x(\cdot) = \{x_1(\cdot), x_2(\cdot)\} = x(\cdot; t_0, x_{10}, x_{20}, \gamma(\cdot))$ *of equation* (1) *such that*

$$\inf_{t \in T} |k(t) + \lambda x_1(t)| \geqslant c > 0.$$

b) *The function* $k(t)$ *is Lebesque measurable and bounded.*

Thus, Problems 1 and 2 may be formulated as follows. In the sequel, a family of partitions

$$\Delta_h = \{\tau_{i,h}\}_{h=0}^{m_h}, \quad \tau_{i+1,h} = \tau_{i,h} + \delta(h), \quad \tau_{0,h} = t_0, \quad \tau_{m_h,h} = \vartheta, \tag{5}$$

of the interval T is assumed to be fixed.

Problem 1. It is required to indicate a rule of choosing of controls u_i^h at the moments τ_i being a mapping of the form

$$U^h : \{\tau_i, \xi_{i-1}^h, \xi_i^h\} \to u_i^h \in \mathbb{R} \tag{6}$$

such that the convergence

$$\int_{t_0}^{\vartheta} |u^h(t) - x_2(t)|^2\,dt \to 0 \tag{7}$$

takes place as h tends to 0. Here, $u^h(t) = u_i^h$ for $t \in \delta_{i,h} = [\tau_{i,h}, \tau_{i+1,h})$.

Problem 2. It is required to indicate a rule of choosing of controls v_i^h at the moments τ_i being a mapping of the form

$$V^h : \{\tau_i, \xi_{i-1}^h, \xi_i^h\} \to v_i^h \in \mathbb{R} \tag{8}$$

such that the convergence

$$\int_{t_0}^{\vartheta} |v^h(t) - \gamma(t)|^2\,dt \to 0 \tag{9}$$

takes place as h tends to 0. Here, $v^h(t) = v_i^h$ for $t \in \delta_{h,i}$.

The analogous problem is considered in the paper [3, 4], where solving algorithms based on the method of controlled models [5–12] are presented. In this paper, we design solving algorithms without the usage of such models.

2. Solving algorithms

Further, we assume that the constants d_1, d_2, k, and γ are such that

$$|x_1(t)| \leqslant d_1, \quad |x_2(t)| \leqslant d_2, \tag{10}$$

$$|\dot{k}(t)| \leqslant k, \quad |\gamma(t)| \leqslant \gamma \quad \text{for a.a.} \quad t \in T. \tag{11}$$

In virtue of (10) and (11), the following inequalities

$$|k(t)| \leqslant k_1 = |k(t_0)| + k(\vartheta - t_0), \tag{12}$$

$$|\dot{x}_1(t)| \leqslant d_3 = k_1 d_2 + \lambda d_1 d_2 + \nu d_1, \tag{13}$$

$$|\dot{x}_2(t)| \leqslant d_4 = k_1 d_2 + \lambda d_1 d_2 + \mu d_2 + \gamma \tag{14}$$

are fulfilled.

In turn, for $t \in [\tau_{i-1}, \tau_i]$, the inequality

$$|x_1(t) - \xi_{i-1}^h| \leqslant h + \int_{\tau_{i-1}}^{t} |\dot{x}_1(\tau)|\,d\tau \leqslant h + d_3\delta \tag{15}$$

holds. Let

$$P_T(\cdot) = \{u(\cdot) \in L_2(T;\mathbb{R})\colon |u(t)| \leqslant d_2 \text{ for a.a. } t \in T\},$$

$$U(x_1(\cdot)) = \{u(\cdot) \in P_T(\cdot)\colon \dot{x}_1(t) = (k(t) + \lambda x_1(t))u(t) - \nu x_1(t) \text{ for a.a. } t \in T\}.$$

Introduce a family of sets

$$U^h(\cdot) = \Big\{u(\cdot) \in P_T(\cdot)\colon$$
$$u(t) = u_i \text{ for a.a. } t \in [\tau_{i-1}, \tau_i),\ u_i \in \widetilde{U}^h(\tau_i, \xi^h_{i-1}, \xi^h_i),\ i \in [1:m]\Big\},$$

where $\tau_i = \tau_{i,h}$, $m = m_h$,

$$\widetilde{U}^h(\tau_i, \xi^h_{i-1}, \xi^h_i) = \Big\{u \in \mathbb{R}\colon$$
$$|u| \leqslant d_2, |(k(\tau_{i-1}) + \lambda \xi^h_{i-1})u - \nu \xi^h_{i-1} - (\xi^h_i - \xi^h_{i-1})\delta^{-1}| \leqslant \sigma^{(1)}_h\Big\},$$
$$\sigma^{(1)}_h = 2h\delta^{-1} + K_1\delta + K_2 h, \quad K_1 = \nu d_3 + (k + \lambda d_3)d_2, \quad K_2 = \nu + \lambda d_2.$$

Lemma 1 *Let $u(\cdot) \in U(x_1(\cdot))$. Then, the inclusion*

$$\delta^{-1}\int_{\tau_{i-1}}^{\tau_i} u(t)\,dt \in \widetilde{U}^h(\tau_i, \xi^h_{i-1}, \xi^h_i), \quad i \in [1:m]$$

takes place.

Proof. Let $f_1(t, x_1, x_2) = (k(t) + \lambda x_1)x_2 - \nu x_1$. Then, for all $q \in \mathbb{R}$, $|q| \leqslant d_2$, and $t \in [\tau_{i-1}, \tau_i]$, by (10), (11), and (15), we obtain

$$|f_1(t, x_1(t), q) - f_1(\tau_{i-1}, \xi^h_{i-1}, q)| \leqslant |k(t) - k(\tau_{i-1})||q| \tag{16}$$
$$+ \lambda|x_1(t) - \xi^h_{i-1}||q| + \nu|x_1(t) - \xi^h_{i-1}| \leqslant k\delta|q| + \lambda(h + d_3\delta)|q| + \nu(h + d_3\delta)$$
$$\leqslant \nu(h + d_3\delta) + \{\lambda h + (k + \lambda d_3)\delta\}d_2 = K_1\delta + K_2 h.$$

In this case, for any $u(\cdot) \in U(x_1(\cdot))$, the inequality

$$\left|\delta^{-1}\int_{\tau_{i-1}}^{\tau_i} f_1(t, x_1(t), u(t))\,dt - \delta^{-1}\int_{\tau_{i-1}}^{\tau_i} f_1(\tau_{i-1}, \xi^h_{i-1}, u(t))\,dt\right|$$
$$\leqslant K_1\delta + K_2 h, \quad i \in [1:m], \tag{17}$$

is fulfilled. In turn, from

$$\delta^{-1}\int_{\tau_{i-1}}^{\tau_i} f_1(t, x_1(t), u(t))\,dt = \delta^{-1}(x_1(\tau_i) - x_1(\tau_{i-1})),$$

we derive

$$\left|\frac{\xi^h_i - \xi^h_{i-1}}{\delta} - \delta^{-1}\int_{\tau_{i-1}}^{\tau_i} f_1(t, x_1(t), u(t))\,dt\right| \leqslant 2\frac{h}{\delta}. \tag{18}$$

The statement of the lemma follows from (17) and (18). The lemma is proved. □

Lemma 2 *For any $t_1 \in T$ and $u(\cdot) \in U^h(\cdot)$, the inequality*

$$\Big|\int_{t_0}^{t_1} \{k(t)+\lambda x_1(t)\}(x_2(t)-u(t))\,dt\Big| \leqslant c_1 h + c_2\delta + c_3 h\delta^{-1}$$

is fulfilled. Here,

$$c_1 = 2 + (\vartheta - t_0)(\nu + \lambda d_2 + K_2),$$

$$c_2 = k_1 + d_1 + d_3 + \lambda(1+d_1)d_2 + (\vartheta - t_0)(\nu d_3 + K_1 + k + \lambda d_2 d_3), \quad c_3 = 2(\vartheta - t_0).$$

Proof. Note that the following equality

$$\int_{t_0}^{t_1} \{k(t)+\lambda x_1(t)\}x_2(t)\,dt = x_1(t_1) - x_1(t_0) + \nu\int_{t_0}^{t_1} x_1(t)\,dt \tag{19}$$

is valid. Let $q = \max\{\tau_i \in \Delta : \tau_i \leqslant t_1\}$. Then, in virtue of (10) and (13), the inequalities

$$|x_1(t_1) - x_1(q)| \leqslant d_3\delta, \tag{20}$$

$$\Big|\int_{t_0}^{t_1} x_1(t)\,dt - \int_{t_0}^{q} x_1(t)\,dt\Big| \leqslant d_1\delta \tag{21}$$

hold. In this case, from (19), (20), and (21) we derive

$$\Big|\int_{t_0}^{t_1} \{k(t)+\lambda x_1(t)\}x_2(t)\,dt - \Big\{x_1(q) - x_1(t_0) + \nu\int_{t_0}^{q} x_1(t)\,dt\Big\}\Big| \leqslant (d_1+d_3)\delta. \tag{22}$$

Using the inclusion $u(\cdot) \in U^h(\cdot)$, we have

$$\Big|\sum_{i=1}^{q}\int_{\tau_{i-1}}^{\tau_i} (k(\tau_{i-1}) + \lambda\xi_{i-1}^h)u_i\,dt - \nu\sum_{i=1}^{q}\delta\xi_{i-1}^h - \xi_q^h + \xi_0^h\Big| \leqslant \delta\sum_{i=1}^{q}\sigma_h^{(1)} \tag{23}$$

$$\leqslant (\vartheta - t_0)(2h\delta^{-1} + K_1\delta + K_2 h).$$

By taking into account (15), we obtain the inequality

$$\Big|\Big\{x_1(q) - x_1(t_0) + \nu\int_{t_0}^{q} x_1(t)\,dt\Big\} - \Big\{\nu\sum_{i=1}^{q}\delta\xi_{i-1}^h - \xi_q^h + \xi_0^h\Big\}\Big| \tag{24}$$

$$\leqslant 2h + \nu\Big|\sum_{i=1}^{q}\int_{\tau_{i-1}}^{\tau_i}\{x_1(t) - \xi_{i-1}^h\}\,dt\Big| \leqslant 2h + \nu(\vartheta - t_0)(h + d_3\delta).$$

Combining (22)–(24), we deduce that

$$\Big|\int_{t_0}^{t_1}(k(t)+\lambda x_1(t))x_2(t)\,dt-\sum_{i=1}^{q}\int_{\tau_{i-1}}^{\tau_i}(k(\tau_{i-1})+\lambda\xi^h_{i-1})u_i\,dt\Big| \tag{25}$$

$$\leqslant \{2+(\vartheta-t_0)(K_2+\nu)\}h+2(\vartheta-t_0)h\delta^{-1}+\{d_1+d_3+(\vartheta-t_0)(\nu d_3+K_1)\}\delta.$$

Then, we have (see (10) and (12))

$$\Big|\int_{\tau_q}^{t_1}(k(\tau_q)+\lambda\xi^h_q)u_q\,dt\Big|\leqslant\{k_1+\lambda(d_1+1)d_2\}\delta. \tag{26}$$

Using (11), (25), and (15), we derive

$$\Big|\int_{t_0}^{t_1}(k(t)+\lambda x_1(t))u(t)\,dt-\sum_{i=1}^{q}\int_{\tau_{i-1}}^{\tau_i}(k(\tau_{i-1})+\lambda\xi^h_{i-1})u_i\,dt\Big|$$

$$\leqslant (k_1+\lambda(d_1+1))d_2\delta+\Big|\sum_{i=1}^{q}\int_{\tau_{i-1}}^{\tau_i}\{k(t)-k(\tau_{i-1})+\lambda(x_1(t)-\xi^h_{i-1})\}u_i\,dt\Big|$$

$$\leqslant (k_1+\lambda(d_1+1)d_2\delta+\delta\sum_{i=1}^{q}\{\delta k+\lambda(h+\delta d_3)d_2\} \tag{27}$$

$$\leqslant (k_1+\lambda(d_1+1)d_2\delta+(\vartheta-t_0)\{(k+\lambda d_2d_3)\delta+\lambda d_2h\}$$

$$=\{k_1+\lambda(d_1+1)d_2+(\vartheta-t_0)(k+\lambda d_2d_3)\}\delta+(\vartheta-t_0)\lambda d_2h.$$

The statement of the lemma follows from (25) and (27). The lemma is proved. □

Lemma 3 ([6, p. 47]) *Let* $u(\cdot)\in L_\infty(T_*;\mathbb{R})$ *and* $v(\cdot)\in W(T_*;\mathbb{R})$, $T_*=[a,b]$, $-\infty<a<b<+\infty$,

$$\Big|\int_a^t u(\tau)\,d\tau\Big|\leqslant\varepsilon,\quad |v(t)|\leqslant K\quad \forall\, t\in T_*.$$

Then, for all $t\in T_*$, *the inequality*

$$\Big|\int_a^t u(\tau)v(\tau)\,d\tau\Big|\leqslant\varepsilon(K+\mathrm{var}(T_*;v(\cdot)))$$

is valid.

Here, the symbol $\mathrm{var}(T_*; v(\cdot))$ means the variation of the function $v(\cdot)$ over the segment T_*, and the symbol $W(T_*;\mathbb{R})$ means the set of functions $y(\cdot) : T_* \to \mathbb{R}$ of bounded variation.

Note that from Condition 1a) it follows that there exists a number $E > 0$ such that

$$\mathrm{var}(T; (k(t) + \lambda x_1(t))^{-1} x_2(t)) \leqslant E.$$

Let

$$U^h(\tau_i, \xi^h_{i-1}, \xi^h_i) = u^h_i = \arg\min\{|v| \colon |v| \in \widetilde{U}^h(\tau_i, \xi^h_{i-1}, \xi^h_i)\} \text{ for } i \in [1 : m_h] \tag{28}$$

and

$$u^h(t) = u^h_i \ \text{ for } \ t \in \delta_{i-1,h}, \ \ i \in [1 : m_h]. \tag{29}$$

Theorem 1 *Let Condition* 1a) *be fulfilled. Then, the following estimate is valid:*

$$|u^h(\cdot) - x_2(\cdot)|^2_{L_2(T;\mathbb{R})} \leqslant (c_1 h + c_2\delta + c_3 h\delta^{-1})(d_2 c^{-1} + E).$$

Proof. Using (28), we deduce for $i \in [1 : m_h]$ that

$$|u^h_i| \leqslant \delta^{-1} \int\limits_{\tau_{i-1}}^{\tau_i} |x_2(t)|\, dt. \tag{30}$$

Therefore, from (29) and (30), for $i \in [1 : m_h]$, we also have

$$\int\limits_{\tau_{i-1}}^{\tau_i} |u^h(t)|^2\, dt \leqslant \int\limits_{\tau_{i-1}}^{\tau_i} |x^h_2(t)|^2\, dt.$$

Therefore, $|u^h(\cdot)|^2_{L_2(T;\mathbb{R})} \leqslant |x_2(\cdot)|^2_{L_2(T;\mathbb{R})}$. From this inequality, it follows that

$$\begin{aligned} |u^h(\cdot) - x_1(\cdot)|^2_{L_2(T;\mathbb{R})} &\leqslant 2|x_2(\cdot)|^2_{L_2(T;\mathbb{R})} - 2(u^h(\cdot), x_2(\cdot))^2_{L_2(T;\mathbb{R})} \\ &= 2(x_2(\cdot) - u^h(\cdot), x_2(\cdot))_{L_2(T;\mathbb{R})}. \end{aligned} \tag{31}$$

Thus, taking into account (31), we obtain

$$|u^h(\cdot) - x_2(\cdot)|^2_{L_2(T;\mathbb{R})}$$

$$\leqslant 2\int\limits_0^{\vartheta} \{(k(t) + \lambda x_1(t))(u^h(\tau) - x_2(\tau))\}\{(k(t) + \lambda x_1(t))^{-1} x_2(t)\}\, d\tau.$$

Then, in virtue of Condition 1a), the following relation

$$|(k(t) + \lambda x_1(t))^{-1} x_2(t)| \leqslant d_2 c^{-1}$$

holds. The statement of the theorem follows from last two inequalities and Lemmas 2 and 3. The theorem is proved. □

On the base of Theorem 1, we conclude that the following theorem can be proved.

Theorem 2 *The mapping U^h of form* (28) *solves Problem 1.*

Thus, Problem 1 is solved. Let us turn to the solution of the second problem. Note that the inequalities

$$|\xi_{1i}^h - x_1(\tau_i)| \leqslant h, \tag{32}$$

$$|\xi_{2i}^h - x_2(\tau_i)| \leqslant h, \tag{33}$$

follows from (2) and (4) . Let

$$Q_T(\cdot) = \{v(\cdot) \in L_2(T;\mathbb{R}) \colon |v(t)| \leqslant \gamma \text{ for a.a. } t \in T\}$$

and

$$V(x(\cdot)) = \Big\{ v(\cdot) \in Q_T(\cdot) \colon$$

$$\dot{x}_2(t) + k(t)x_2(t) + (\lambda x_1(t) + \mu)x_2(t) = v(t) \text{ for a.a. } t \in T \Big\}.$$

Introduce the family of sets

$$\widetilde{V}^h(\tau_i, \xi_{i-1}^h, \xi_i^h) = \Big\{ v \in \mathbb{R} \colon \tag{34}$$

$$|v| \leqslant \gamma,\ |v - k(\tau_{i-1})\xi_{2i-1}^h - (\lambda\xi_{1i-1}^h + \mu)\xi_{2i-1}^h - (\xi_{2i}^h - \xi_{2i-1}^h)\delta^{-1}| \leqslant \sigma_h^{(2)} \Big\},$$

$$V^h(\cdot) = \Big\{ v(\cdot) \in Q_T(\cdot) \colon$$

$$v(t) = v_i \text{ for a.a. } t \in [\tau_{i-1}, \tau_i),\ \ v_i \in \widetilde{V}^h(\tau_i, \xi_{i-1}^h, \xi_i^h),\ \ i \in [1:m] \Big\},$$

where

$$K_3 = k_1 + \mu + \lambda(1 + d_1 + d_2), \quad K_4 = kd_2 + (k_1 + \mu)d_4 + \lambda(d_2 d_3 + d_4 + d_1 d_4)\delta,$$

$$\sigma_h^{(2)} = K_3 h + K_4 \delta + 2h\delta^{-1}.$$

Let, in addition,

$$V^h(\tau_i, \xi_{i-1}^h, \xi_i^h) = v_i^h = \arg\min\{|v| \colon |v| \in \widetilde{V}^h(\tau_i, \xi_{i-1}^h, \xi_i^h)\}, \quad i \in [1:m_h] \tag{35}$$

and

$$v^h(t) = v_{i-1}^h \quad \text{for} \quad t \in \delta_{i-1,h}, \quad i \in [1:m_h].$$

Lemma 4 *Let $v(\cdot) \in V(x(\cdot))$. Then, the inclusions*

$$\delta^{-1} \int_{\tau_{i-1}}^{\tau_i} v(t)\,dt \in \widetilde{V}^h(\tau_i, \xi_{i-1}^h, \xi_i^h), \quad i \in [1:m],$$

take place.

Proof. Let $f_2(t,x_1,x_2) = -k(t)x_2 - (\lambda x_1 + \mu)x_2$. Then, for all $t \in [\tau_{i-1}, \tau_i]$, the inequality

$$\Phi_i(t) \equiv |f_2(t,x_1(t),x_2(t)) - f_2(\tau_{i-1},\xi^h_{1i-1},\xi^h_{2i-1})| \leqslant I^{(1)}_{i-1}(t) + I^{(2)}_{i-1}(t) + I^{(3)}_{i-1}(t)$$

is valid. Here,

$$I^{(1)}_{i-1}(t) = |k(t) - k(\tau_{i-1})||x_2(t)|,$$
$$I^{(2)}_{i-1}(t) = |k(\tau_{i-1})||x_2(t) - \xi^h_{2i-1}| + \mu|x_2(t) - \xi^h_{2i-1}|,$$
$$I^{(3)}_{i-1}(t) = \lambda|x_1(t)x_2(t) - \xi^h_{1i-1}\xi^h_{2i-1}|.$$

In virtue of (10) and (11), the following inequality

$$I^{(1)}_{i-1}(t) \leqslant kd_2\delta \tag{36}$$

holds. In turn, from (33) and (14), we derive for $t \in [\tau_{i-1}, \tau_i]$ the inequality

$$|x_2(t)) - \xi^h_{2i-1})| \leqslant h + d_4\delta. \tag{37}$$

Therefore, from (12) and (37), we get the estimate

$$I^{(2)}_{i-1}(t) \leqslant (k_1 + \mu)(h + d_4\delta). \tag{38}$$

Note that

$$|\xi^h_{1i-1}| \leqslant 1 + d_2. \tag{39}$$

Using (10), (32), (37), and (39), we derive

$$\begin{aligned} I^{(3)}_{i-1}(t) &= \lambda|x_1(t)x_2(t) - \xi^h_{1i-1}\xi^h_{2i-1}| \\ &\leqslant \lambda|(x_1(t) - \xi^h_{1i-1})x_2(t)| + \lambda|\xi^h_{1i-1}(x_2(t) - \xi^h_{2i-1})| \\ &\leqslant \lambda d_2(h + d_3\delta) + \lambda(1 + d_1)(h + d_4\delta) = \lambda(1 + d_1 + d_2)h + \lambda(d_2d_3 + d_4 + d_1d_4)\delta. \end{aligned} \tag{40}$$

Combining (36), (38), and (40), we deduce that

$$\Phi_i(t) \leqslant K_4\delta + K_3h. \tag{41}$$

Also, from the equality

$$\delta^{-1}\int\limits_{\tau_{i-1}}^{\tau_i} f_2(t,x_1(t),x_2(t))\,dt = \delta^{-1}(x_2(\tau_i) - x_2(\tau_{i-1})), \tag{42}$$

we have

$$\left|\delta^{-1}\int\limits_{\tau_{i-1}}^{\tau_i} f_2(t,x_1(t),x_2(t))\,dt - \frac{\xi^h_{2i} - \xi^h_{2i-1}}{\delta}\right| \leqslant 2\frac{h}{\delta}.$$

In this case, from (41) and 42), it follows that the inequalities

$$\left|\delta^{-1}\int\limits_{\tau_{i-1}}^{\tau_i} v(t)\,dt - k(\tau_{i-1})\xi^h_{2i-1} - (\lambda\xi^h_{i-1}+\mu)\xi^h_{2i-1} - (\xi^h_{2i}-\xi^h_{2i-1})\delta^{-1}\right| \leqslant \sigma^{(2)}_h,$$

$$i\in[1:m],$$

are valid for any $v(\cdot)\in V(x(\cdot))$. The lemma is proved. □

Lemma 5 *Let sequences* $\{h_j\}_{j=1}^{\infty}\subset\mathbb{R}$ *and* $\{v_j(\cdot)\}_{j=1}^{\infty}\subset Q_T(\cdot)$ *with the properties*

$$h_j\delta^{-1}(h_j)\to 0 \quad \textit{as } j\to\infty, \tag{43}$$

$$v_j(\cdot)\in V^{h_j}(\cdot),\quad v_j(\cdot)\to v_0(\cdot) \quad \textit{weakly in } L_2(T;\mathbb{R}) \textit{ as } j\to\infty \tag{44}$$

be given. Then, $v_0(\cdot)\in V(x(\cdot))$.

Proof. Assume the contrary, i.e., $v_0(\cdot)\notin V(x(\cdot))$. Then, there exist $t_1,t_2\in T$, $t_1<t_2$ such that

$$\left|\int\limits_{t_1}^{t_2}(k(t)+\lambda x_1(t)+\mu)x_2(t)\,dt + x_2(t_2)-x_2(t_1)-\int\limits_{t_1}^{t_2} v_0(t)\,dt\right| = b>0. \tag{45}$$

Let j_1 be such that, for all $j\geqslant j_1$ and all $t_*,t^*\in T$ with the properties $0\leqslant t^*-t_*\leqslant\delta(h_j)$, the inequalities

$$\left|\int\limits_{t_*}^{t^*}(k(t)+\lambda x_1(t)+\mu)x_2(t)\,dt\right|\leqslant b/8,\qquad \left|x_2(t_*)-x_2(t^*)-\int\limits_{t_*}^{t^*} v_0(t)\,dt\right|\leqslant b/8 \tag{46}$$

take place. Denote $p_j=\min\{t\in\Delta_{h_j}\colon t\leqslant t_2\}$ and $q_j=\max\{t\in\Delta_{h_j}\colon t\leqslant t_2\}$. From (45) and (46), we have

$$\left|\int\limits_{p_j}^{q_j}\{(k(t)+\lambda x_1(t)+\mu)x_2(t)+\dot{x}_2(t)-v_0(t)\}\,dt\right|\geqslant b/2 \quad \text{for} \quad \mathrm{j}\geqslant j_1.$$

In this case, for $j\geqslant j_1$, we obtain

$$\sum_{q=1}^{3} I_j^{(q)}\geqslant b/2, \tag{47}$$

where

$$I_j^{(1)}=\left|\int\limits_{p_j}^{q_j}(v_j(t)-v_0(t))\,dt\right|,$$

$$I_j^{(2)} = \Big| \int_{p_j}^{q_j} \{(k(t) + \lambda x_1(t) + \mu) x_2(t) + \dot{x}_2(t)\}\, dt$$

$$- \sum_{i=p_j}^{q_{j-1}} \int_{\tau_i}^{\tau_{i+1}} \{k(\tau_{i-1})\xi_{2i-1}^{h_j} + (\lambda \xi_{1i-1}^{h_j} + \mu)\xi_{2i-1}^{h_j} + (\xi_{2i}^{h_j} - \xi_{2i-1}^{h_j})\delta^{-1}(h_j)\}\, dt \Big|,$$

$$I_j^{(3)} = \Big| \sum_{i=p_j}^{q_{j-1}} \int_{\tau_i}^{\tau_{i+1}} \{k(\tau_{i-1})\xi_{2i-1}^{h_j} + (\lambda \xi_{1i-1}^{h_j} + \mu)\xi_{2i-1}^{h_j} + (\xi_{2i}^{h_j} - \xi_{2i-1}^{h_j})\delta^{-1}(h_j)\}\, dt$$

$$- \int_{p_j}^{q_j} v_j(t)\}\, dt \Big|.$$

Then, relation (44) implies the convergence

$$I_j^{(1)} \to 0 \quad \text{as} \quad j \to \infty. \tag{48}$$

In turn, using (41), (15), (33), and (43), we deduce that

$$I_j^{(2)} \to 0 \quad \text{as} \quad j \to \infty. \tag{49}$$

Taking into account the inclusion $v_j(\cdot) \in V^{h_j}(\cdot)$, we obtain

$$v_j(t) = v_{ij} \in \widetilde{V}^{h_j}(\tau_i, \xi_{i-1}^{h_j}, \xi_i^{h_j}) \quad \text{for} \quad t \in [\tau_i, \tau_{i+1}).$$

Consequently,

$$|v_{ij} - \{(k(\tau_{i-1}) + \lambda \xi_{1i-1}^{h_j} + \mu)\xi_{2i-1}^{h_j} + (\xi_{2i}^{h_j} - \xi_{2i-1}^{h_j})\delta^{-1}(h_j)\}| \leqslant \sigma_{h_j}^{(2)}. \tag{50}$$

In this case, from (50), we deduce that

$$\Big| \int_{p_j}^{q_{j-1}} v_j(t)\, dt - \sum_{i=p_j}^{q_{j-1}} \{k(\tau_{i-1})\xi_{2i-1}^{h_j} + (\lambda \xi_{1i-1}^{h_j} + \mu)\xi_{2i-1}^{h_j} + (\xi_{2i}^{h_j} - \xi_{2i-1}^{h_j})\delta^{-1}\} \Big| \tag{51}$$

$$\leqslant (\vartheta - t_0)\sigma_{h_j}^{(2)}.$$

Due to (51), we conclude that

$$I_j^{(3)} \to 0 \quad \text{as} \quad j \to \infty. \tag{52}$$

Relations (48), (49), and (52) contradict (47). The lemma is proved. □

Theorem 3 *Let $\delta(h) \to 0$, $h\delta^{-1}(h) \to 0$ as $h \to 0$. Then, the convergence*

$$v^h(\cdot) \to \gamma(\cdot) \quad \text{in } L_2(T;\mathbb{R}) \text{ as } h \to 0$$

takes place.

Proof. Let

$$\widetilde{v}_h(t) = \begin{cases} v^h(t+\delta) & \text{for} \quad t \in [0, \vartheta - \delta), \\ v^h_{m_h} & \text{for} \quad t \in [\vartheta - \delta, \vartheta]. \end{cases}$$

First, we prove that $\widetilde{v}_h(\cdot) \to \gamma(\cdot)$ as $h \to 0$ in $L_2(T;\mathbb{R})$. To do this, it is sufischently to prove that for any sequence $\{h_j\}_{j=1}^{\infty}$, $h_j > 0$, $h_j \to 0$ as $j \to \infty$, the convergence

$$\widetilde{v}_{h_j}(\cdot) \to \gamma(\cdot) \quad \text{in } L_2(T;\mathbb{R}) \text{ as } j \to \infty$$

takes place. Assuming the contrary, we conclude that there exists a subsequence $\{\widetilde{v}_{h_j}(\cdot)\}_{j=1}^{\infty}$ (for simplicity we denote it by the symbol $\{u_j(\cdot)\}_{j=1}^{\infty}$) such that

$$u_j(\cdot) \to u_0(\cdot) \neq \gamma(\cdot) \quad \text{weakly in } L_2(T;\mathbb{R}) \text{ as } j \to \infty. \tag{53}$$

Let $\delta_j = \delta(h_j)$,

$$\bar{u}_j(t) \equiv u_{j,i} = \delta_j^{-1} \int\limits_{\tau_{i,h_j}}^{\tau_{i+1,h_j}} \gamma(t)\,dt \quad \text{for } t \in [\tau_{i,h_j}, \tau_{i+1,h_j}). \tag{54}$$

In view of Lemma 4, we derive the inclusion $u_{j,i} \in \widetilde{V}^{h_j}(\tau_i, \xi^{h_j}_{i-1}, \xi^{h_j}_i)$. Consequently, $\bar{u}_j(\cdot) \in V^{h_j}(\cdot)$. Due to the rule of choosing the value v^h_i, we have

$$|u_j(\cdot)|_{L_2(T;\mathbb{R})} \leqslant |\bar{u}_j(\cdot)|_{L_2(T;\mathbb{R})}. \tag{55}$$

Using (54), we deduce that

$$|\bar{u}_j(\cdot)|^2_{L_2([\tau_i,\tau_{i+1});\mathbb{R})} = \delta_j |u_{j,i}|^2 \leqslant \delta_j^{-1} \Big(\int\limits_{\tau_i}^{\tau_{i+1}} |\gamma(t)|\,dt \Big)^2$$

$$\leqslant \int\limits_{\tau_i}^{\tau_{i+1}} |\gamma(t)|^2\,dt = |\gamma(\cdot)|^2_{L_2([\tau_i,\tau_{i+1});\mathbb{R})}.$$

Therefore,

$$|\bar{u}_j(\cdot)|_{L_2(T;\mathbb{R})} \leqslant |\gamma(\cdot)|_{L_2(T;\mathbb{R})}. \tag{56}$$

Combining (55) and (56), we conclude that

$$|u_j(\cdot)|_{L_2(T;\mathbb{R})} \leqslant |\gamma(\cdot)|_{L_2(T;\mathbb{R})}.$$

Consequently,

$$\limsup_{j\to\infty} |u_j(\cdot)|_{L_2(T;\mathbb{R})} \leqslant |\gamma(\cdot)|_{L_2(T;\mathbb{R})}. \tag{57}$$

As well, in virtue of the known property of the weak limit, we obtain

$$\liminf_{j\to\infty} |u_j(\cdot)|_{L_2(T;\mathbb{R})} \geqslant |u_0(\cdot)|_{L_2(T;\mathbb{R})}.$$

This and (57) implies

$$|u_0(\cdot)|_{L_2(T;\mathbb{R})} \leqslant |\gamma(\cdot)|_{L_2(T;\mathbb{R})}.$$

Then, in virtue of Lemma 5, we have $u_0(\cdot) \in V(x(\cdot))$. Therefore, $u_0(\cdot) = \gamma(\cdot)$. This equality contradicts (53). So, $\widetilde{v}_h(\cdot) \to \gamma(\cdot)$ as $h \to 0$ in $L_2(T;\mathbb{R})$. Further, we prove the second statement of the theorem. We have

$$\int_0^{\vartheta} |v^h(t) - \gamma(t)|^2\, dt$$

$$\leqslant \int_0^{\delta} |\gamma(t) - v_0^h|^2\, dt + 2\int_{\delta}^{\vartheta} |\widetilde{v}_h(t-\delta) - \gamma(t-\delta)|^2\, dt + 2\int_{\delta}^{\vartheta} |\gamma(t-\delta) - \gamma(t)|^2\, dt.$$

Using the relations $\gamma(\cdot) \in L_2(T;\mathbb{R})$ and $|v_0^h| \leqslant \gamma$, we deduce that the second and third terms in the right-hand part of last inequality tend to zero as $\delta \to 0$. The second term does not exceed the value $2|\widetilde{v}_h(\cdot) - \gamma(\cdot)|^2_{L_2(T;\mathbb{R})}$, that is infinitely small as $h \to 0$, in virtue of the proved convergence. The theorem is proved. □

From Theorem 3, we conclude that the following theorem takes place.

Theorem 4 *The mapping V^h of form* (35) *solves Problem 2.*

3. Conclusions

In the paper, two algorithms of stable reconstruction of unknown characteristics of dynamical system described by second order nonlinear systems of ordinary differential equations are specified. These algorithms are stable with respect to informational noises and computational errors. The algorithms are based on constructions of the theory of dynamic inversion.

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