Minimum energy control of positive 2D continuous-discrete linear systems with bounded inputs

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A new formulation of the minimum energy control problem for the positive 2D continuous-discrete linear systems with bounded inputs is proposed. Necessary and sufficient conditions for the reachability of the systems are established. Conditions for the existence of the solution to the minimum energy control problem and a procedure for computation of an input minimizing the given performance index are given. Effectiveness of the procedure is demonstrated on numerical example.

Key words: 2D continuous-discrete, linear, positive, reachability, minimum energy control.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [5, 15]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The positive 2D continuous-discrete linear systems have been introduced in [19], positive hybrid linear systems in [16] and the positive fractional 2D hybrid systems in [18]. Different methods of solvability of 2D hybrid linear systems have been discussed in [30] and the solution to singular 2D hybrid linear systems has been derived in [32]. The realization problem for positive 2D hybrid systems has been addressed in [20]. Some problems of dynamics and control of 2D hybrid systems have been considered in [4, 6]. The problems of stability and robust stability of 1D and 2D continuous-discrete linear systems have been investigated in [1-3, 14, 31, 33, 34] and of positive fractional 2D continuous-discrete linear systems in [17]. Recently the stability and robust stability of general model and of Roesser type model of scalar continuous-discrete linear systems

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have been analyzed by Busşłowicz in [2, 3]. Stability of continuous-discrete 2D linear systems has been considered in [22]. The minimum energy control problem for standard linear systems has been formulated and solved by J. Klamka [24-29] and for 2D linear systems with variable coefficients in [23]. The controllability and minimum energy control problem of fractional discrete-time linear systems has been investigated by Klamka in [24]. The minimum energy control of fractional positive and standard positive continuous-time linear systems has been addressed in [7, 9], for descriptor positive linear systems in [8, 13] and with bounded inputs in [10-12].

In this paper a new formulation and solution to the minimum energy control problem for positive 2D continuous-discrete linear systems with bounded inputs will be presented. The paper is organized as follows. In section 2 necessary and sufficient conditions for the positivity of 2D continuous-discrete linear systems are established. The reachability and the problem formulation are given in section 3. Problem solution and a procedure for solving the minimum energy control problem are given in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n \times m}$ – the set of $n \times m$ real matrices, $\Re^{n \times m}_+$ – the set of $n \times m$ matrices with nonnegative entries and $\Re^n_+ = \Re^{n \times 1}_+$, M_n – the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n \times n$ identity matrix.

2. Positivity of 2D continuous-discrete systems

Consider the 2D continuous-discrete linear system

$$\dot{x}(t,i) = Ax(t,i) + Bu(t,i) \tag{1}$$

where $\dot{x}(t,i) = \frac{\partial x(t,i)}{\partial t}$, $x(t,i) \in \Re^n$, $u(t,i) \in \Re^m$ are the state and input vectors and $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$ $(n \geqslant m)$ and $t \in \Re_+$ is continuous variable (usually time) and $i \in Z_+ = \{0,1,\dots\}$ is discrete variable.

Definition 7 *The system* (1) *is called (internally) positive if* $x(t,i) \in \Re^n_+$, $t \in \Re_+$, $i \in Z_+$ *for any boundary conditions* $x_{0i} \in \Re^n_+$, $x_{t0} \in \Re^n_+$, $\dot{x}_{t0} \in \Re^n_+$, *and all inputs* $u(t,i) \in \Re^m_+$, $t \in \Re_+$, $i \in Z_+$.

Theorem 3 *The system* (1) *is positive if and only if*

$$A \in M_n \quad \text{and} \quad B \in \Re^{n \times m}_+. \tag{2}$$

Proof *Necessity.* Let $u(t,i) = 0$, $t \geqslant 0$ and $x(0,i) = e_k$ (k-th ($k = 1,\dots,n$) column of the identity matrix I_n). The trajectory does not live the orthant $\Re^n_+$ only if the derivative $\dot{x}(0,i) = Ae_k \geqslant 0$, what implies $a_{ij} \geqslant 0$, $i \neq j$. Therefore, the matrix A has to be the Metzler matrix. For the same reasons for $x(0,i) = 0$ we have $\dot{x}(0,i) = Bu(0,i) \geqslant 0$, what

implies $B \in \mathfrak{R}_+^{n \times m}$, since $u(0,i) \in \mathfrak{R}_+^m$ may be arbitrary for $i \in Z_+$.

Sufficiency. The solution of the equation (1) is given by

$$x(t,i) = e^{At}x(0,i) + \int_0^t e^{A(t-\tau)}Bu(\tau,i)d\tau. \qquad (3)$$

It is well-known [15, 21] that $e^{At} \in \mathfrak{R}_+^{n \times n}$, $t \in \mathfrak{R}_+$ if and only if $A \in M_n$. From (3) it follows that if the conditions (2) are met and $x(0,i) \in \mathfrak{R}_+^n$, $u(t,i) \in \mathfrak{R}_+^m$, $t \in \mathfrak{R}_+$, $i \in Z_+$ then $x(t,i) \in \mathfrak{R}_+^n$, $t \in \mathfrak{R}_+$, $i \in Z_+$. Hence by Definition 1 the system (1) is positive. $\square$

3. Reachability and problem formulation

Consider the positive 2D continuous-discrete linear system (1).

Definition 8 *The positive system* (1) *is called reachable in segment of line* $\{[t_f,0],[t_f,q]\}$ *if for any given final state vector* $x_f \in \mathfrak{R}^n$ *there exists an input* $u(t,i) \in \mathfrak{R}_+^m$, $0 \leqslant t \leqslant t_f$, $0 \leqslant i \leqslant q$ *that steers the state vector* $x(t,i)$ *of the system from* $x(0,i) = 0$, $i = 0,1,\ldots,q$ *to* $x_f = x(t_f,0) + x(t_f,1) + \ldots + x(t_f,q)$.

Theorem 4 *The positive system* (1) *is reachable on the segment of line* $\{[t_f,0],[t_f,q]\}$ *if and only if the matrix* $A \in M_n$ *is diagonal and the matrix* $B \in \mathfrak{R}_+^{n \times m}$ *is monomial.*

Proof Using (3) for $t = t_f$, $i = 0,1,\ldots,q$ and $x(0,i) = 0$, $i = 0,1,\ldots,q$ we obtain

$$x_f = x(t_f,0) + x(t_f,1) + \ldots + x(t_f,q) = \int_0^{t_f} e^{A(t_f-\tau)}\bar{B}\bar{u}(\tau)d\tau \qquad (4a)$$

where

$$\bar{B} = [\ B \quad B \quad \ldots \quad B\ ] \in \mathfrak{R}_+^{n \times \bar{m}},$$

$$\bar{u}(\tau) = \begin{bmatrix} u(\tau,0) \\ u(\tau,1) \\ \vdots \\ u(\tau,q) \end{bmatrix} \in \mathfrak{R}_+^{\bar{m}}, \quad \bar{m} = n(q+1). \qquad (4b)$$

It is well-known [15, 21] that if $A \in M_n$ is diagonal then $e^{At} \in \mathfrak{R}_+^{n \times n}$ is also diagonal and if $B \in \mathfrak{R}_+^{n \times m}$ is monomial then $\bar{B}\bar{B}^T \in \mathfrak{R}_+^{n \times n}$ is also monomial. In this case the matrix

$$R(t_f,q) = \int_0^{t_f} e^{A\tau}\bar{B}\bar{B}^T e^{A^T\tau}d\tau \in \mathfrak{R}_+^{n \times n} \qquad (5)$$

is also monomial and $R^{-1}(t_f, q) \in \Re_+^{n \times n}$. The input

$$\hat{u}(t) = \bar{B}^T e^{A^T(t_f - t)} R^{-1}(t_f, q) x_f \tag{6}$$

steers the state of the system (1) from $x(0, i) = 0$, $i = 0, 1, \ldots, q$ to the segment of line $\{[t_f, 0], [t_f, q]\}$. Using (4) and (3) we obtain

$$
\begin{aligned}
x(t_f, q) &= \int_0^{t_f} e^{A(t_f - \tau)} \bar{B} \hat{u}(\tau) d\tau = \int_0^{t_f} e^{A(t_f - \tau)} \bar{B} \bar{B}^T e^{A^T(t_f - \tau)} d\tau R^{-1}(t_f, q) x_f \\
&= \int_0^{t_f} e^{A\tau} \bar{B} \bar{B}^T e^{A^T \tau} d\tau R^{-1}(t_f, q) x_f = x_f.
\end{aligned} \tag{7}
$$

Necessity. From the Cayley-Hamilton theorem we have

$$e^{At} = \sum_{k=0}^{n-1} c_k(t) A^k \tag{8}$$

where $c_k(t)$, $k = 0, 1, \ldots, n-1$ are some nonzero functions of time depending on the matrix A. Substitution of (8) into

$$\int_0^{t_f} e^{A(t_f - \tau)} \bar{B} \bar{u}(\tau) d\tau \tag{9}$$

yields

$$x_f = \begin{bmatrix} \bar{B} & A\bar{B} & \ldots & A^{n-1}\bar{B} \end{bmatrix} \begin{bmatrix} v_0(t_f) \\ v_1(t_f) \\ \vdots \\ v_{n-1}(t_f) \end{bmatrix} \tag{10}$$

where

$$v_k(t_f) = \int_0^{t_f} c_k(\tau) \bar{u}(t_f - \tau) d\tau, \quad k = 0, 1, \ldots, n-1. \tag{11}$$

For given $x_f \in \Re_+^n$ it is possible to compute nonnegative $v_k(t_f)$, $k = 0, 1, \ldots, n-1$ if and only if the matrix

$$\begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} \tag{12}$$

has n linearly independent monomial columns and this takes place only if the matrix $[A \ B]$ contains n linearly independent monomial columns [15, 21].

Note that for nonnegative $v_k(t_f)$, $k = 0, 1, \ldots, n-1$ it is possible to find a nonnegative input $\bar{u}(t) \in \Re_+^{\bar{m}}$ only if the matrix $B \in \Re_+^{n \times n}$ is monomial and the matrix $a \in M_n$ is diagonal. □

If the positive system (1) is reachable on the segment of line $\{[t_f,0],[t_f,q]\}$, then usually there exists many different inputs $\bar{u}(\tau) \in \Re_+^{\bar{m}}$ that steers the state of the system from $x(0,i) = 0$, $i = 0,1,\dots,q$ to $x_f = x(t_f,0) + x(t_f,1) + \dots + x(t_f,q) \in \Re_+^n$. Among these inputs we are looking for an input $\hat{u}(t) \in \Re_+^{\bar{m}}$ for $t \in [0,t_f]$ that minimizes the performance index

$$I(u) = \int_0^{t_f} u^T(\tau)Qu(\tau)d\tau \tag{13}$$

where $Q \in \Re_+^{\bar{m}\times\bar{m}}$ is a symmetric positive defined matrix and $Q^{-1} \in \Re_+^{\bar{m}\times\bar{m}}$.

The minimum energy control problem for can be stated as follows: Given the matrices $A \in M_n$, $B \in \Re_+^{n\times m}$, $Q \in \Re_+^{\bar{m}\times\bar{m}}$, the number q and $x_f \in \Re_+^n$, find an input $\bar{u}(t) \in \Re_+^{\bar{m}}$ satisfying the condition

$$u(t,i) \leqslant U \in \Re_+^m \quad \text{for} \quad t \in [0,t_f], \ i \in [0,q], \ (U-\text{given}) \tag{14}$$

and minimum t_f that steers the state vector of the system from $x(0,i) = 0$, $i = 0,1,\dots,q$ to $x_f = x(t_f,0) + x(t_f,1) + \dots + x(t_f,q) \in \Re_+^n$ and minimizes the performance index (13).

4. Problem solution

To solve the problem we define the matrix

$$W = W(t_f,Q) = \int_0^{t_f} e^{A(t_f-\tau)}\bar{B}Q^{-1}\bar{B}^T e^{A^T(t_f-\tau)}d\tau \tag{15}$$

where $\bar{B}$ is defined by (4b).

By Theorem 4 the matrix (15) is monomial and $W^{-1} \in \Re_+^{n\times n}$ if and only if the positive system (1) is reachable at the segment of line $\{[t_f,0],[t_f,q]\}$. In this case we may define the input

$$\hat{u}(t) = Q^{-1}\bar{B}^T e^{A^T(t_f-t)}W^{-1}x_f \quad \text{for} \quad t \in [0,t_f]. \tag{16}$$

Note that $\hat{u}(t) \in \Re_+^{\bar{m}}$ for $t \in [0,t_f]$ if

$$Q^{-1} \in \Re_+^{\bar{m}\times\bar{m}} \quad \text{for any} \quad x_f \in \Re_+^n \quad \text{and} \quad W^{-1}x_f \in \Re_+^{n\times n}. \tag{17}$$

Theorem 5 *Let $\bar{u}(t) \in \Re_+^{\bar{m}}$ for $t \in [0,t_f]$ be an input satisfying (14) that steers the state of the positive system (1) from $x(0,i) = 0$, $i = 0,1,\dots,q$ to $x_f = x(t_f,0) + x(t_f,1) + \dots + x(t_f,q) \in \Re_+^n$. Then the input (16) satisfying (14) also steers the state of the system from $x(0,i) = 0$, $i = 0,1,\dots,q$ to $x_f = x(t_f,0) + x(t_f,1) + \dots + x(t_f,q) \in \Re_+^n$ and minimizes the performance index (13), i.e. $I(\hat{u}) \leqslant I(\bar{u})$. The minimal value of the performance index (13) is equal to*

$$I(\hat{u}) = x_f^T W^{-1} x_f. \tag{18}$$

Proof If the conditions (17) are met then $\hat{u}(t) \in \mathfrak{R}_+^{\bar{m}}$ for $t \in [0, t_f]$. We shall show that the input satisfying (14) steers the state of the system from $x(0, i) = 0$, $i = 0, 1, \ldots, q$ to $x_f \in \mathfrak{R}_+^n$. Substitution of (16) into (4a) for $t = t_f$ yields

$$x(t_f) = \int\limits_0^{t_f} e^{A(t_f - \tau)} \bar{B} \hat{u}(\tau) d\tau = \int\limits_0^{t_f} e^{A(t_f - \tau)} \bar{B} Q^{-1} \bar{B}^T e^{A^T(t_f - \tau)} d\tau W^{-1} x_f = x_f \quad (19)$$

since (15) holds. By assumption the inputs $\bar{u}(t)$ and $\hat{u}(t)$, $t \in [0, t_f]$ satisfying (14) steers the state of the system from $x(0, i) = 0$, $i = 0, 1, \ldots, q$ to $x_f \in \mathfrak{R}_+^n$. Hence

$$x_f = \int\limits_0^{t_f} e^{A(t_f - \tau)} \bar{B} \bar{u}(\tau) d\tau = \int\limits_0^{t_f} e^{A(t_f - \tau)} \bar{B} \hat{u}(\tau) d\tau \quad (20a)$$

$$\int\limits_0^{t_f} e^{A(t_f - \tau)} \bar{B} [\bar{u}(\tau) - \hat{u}(\tau)] d\tau = 0. \quad (20b)$$

By transposition of (20b) and postmultiplication by $W^{-1} x_f$ we obtain

$$\int\limits_0^{t_f} [\bar{u}(\tau) - \hat{u}(\tau)]^T \bar{B}^T e^{A^T(t_f - \tau)} d\tau W^{-1} x_f = 0. \quad (21)$$

Substitution of (16) into (21) yields

$$\int\limits_0^{t_f} [\bar{u}(\tau) - \hat{u}(\tau)]^T Q \hat{u}(\tau) = 0 \quad (22)$$

since

$$Q \hat{u}(\tau) = \bar{B}^T e^{A^T(t_f - \tau)} W^{-1} x_f = x_f. \quad (23)$$

Using (22) it is easy to verify that

$$\int\limits_0^{t_f} \bar{u}(\tau)^T Q \bar{u}(\tau) d\tau = \int\limits_0^{t_f} \hat{u}(\tau)^T Q \hat{u}(\tau) d\tau + \int\limits_0^{t_f} [\bar{u}(\tau) - \hat{u}(\tau)]^T Q [\bar{u}(\tau) - \hat{u}(\tau)] d\tau. \quad (24)$$

From (24) it follows that $I(\hat{u}) < I(\bar{u})$ since the second term in the right-hand side of inequality is nonnegative.

To find the minimal value of the performance index (13) we substitute (16) into (13) and we obtain

$$I(\hat{u}) = \int\limits_0^{t_f} \hat{u}^T(\tau) Q \hat{u}(\tau) d\tau = x_f^T W^{-1} \int\limits_0^{t_f} e^{A(t_f - \tau)} \bar{B} Q^{-1} \bar{B}^T e^{A^T(t_f - \tau)} d\tau W^{-1} x_f = x_f^T W^{-1} x_f$$

$$(25)$$

since (15) holds. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Theorem 6 *If the diagonal matrix Q is a scalar matrix*

$$Q = \mathrm{diag}[q_1, ..., q_1] \in \mathfrak{R}_+^{\bar{n} \times \bar{n}} \tag{26}$$

then the input (16) *is independent of Q and has the form*

$$\hat{u}(t) = \bar{B}^T e^{A^T(t_f - t)} \left[\int_0^{t_f} e^{A\tau} \bar{B} \bar{B}^T e^{A^T \tau} d\tau \right] x_f \in \mathfrak{R}_+^{\bar{n}} \tag{27}$$

for any $x_f \in \mathfrak{R}_+^n$.

Proof If (26) holds then from (15) we have

$$W = \frac{1}{q_1} \int_0^{t_f} e^{A\tau} \bar{B} \bar{B}^T e^{A^T \tau} d\tau \tag{28}$$

and from (16)

$$\hat{u}(t) = Q^{-1} \bar{B}^T e^{A^T(t_f - t)} W^{-1} x_f = \bar{B}^T e^{A^T(t_f - t)} \left[\int_0^{t_f} e^{A\tau} \bar{B} \bar{B}^T e^{A^T \tau} d\tau \right]^{-1} x_f \in \mathfrak{R}_+^{\bar{n}} \tag{29}$$

for any $x_f \in \mathfrak{R}_+^n$.   □

From (16) we have

$$\frac{d\hat{u}(t)}{dt} = -EA^T e^{A^T(t_f - t)} F \tag{30a}$$

where

$$E = Q^{-1} B^T, \quad F = W^{-1} x_f. \tag{30b}$$

Using (30) we may find $t \in [0, t_f]$ for which $\hat{u}(t) \in \mathfrak{R}_+^n$ reaches its maximal value. Note that if all eigenvalues of the matrix A have positive real parts then $\hat{u}(t)$ reaches its maximal value for $t = 0$ and if they have negative real parts then for $t = t_f$.

From the above considerations we have the following procedure for computation the optimal inputs satisfying the condition (14) that steers the state of the system from $x_0 = 0$ to $x_f \in \mathfrak{R}_+^n$ and minimize the performance index (13).

Procedure 1

Step 1. Knowing $A \in M_n$ compute e^{At}.
Step 2. Using (15) compute the matrix W knowing the matrices A, B, Q for some t_f.
Step 3. Using (16) and (30) compute the input (16) and t_f satisfying the condition (14) for given $U \in \mathfrak{R}_+^n$ and $x_f \in \mathfrak{R}_+^n$.
Step 4. Using (18) compute the minimal value of the performance index.

Example 1 Consider the positive system (1) with matrices

$$
A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{31}
$$

and the performance index (13) with the matrix

$$
Q = \mathrm{diag}[2,2,2,2,2,2,2,2,2] \tag{32}
$$

satisfying (14) for

$$
u = [1] \tag{33}
$$

and $q = 2$. Compute the optimal input $\hat{u}(t) \in \mathfrak{R}_+^9$ for $t \in [0, t_f]$ and t_f is minimal that steers the state of the system from $x(0,i) = 0$, $i = 0, 1, 2$ to $x_f = [\ 1 \quad 1 \quad 1 \ ]^T \in \mathfrak{R}_+^3$ (T denote the transpose). By Theorem 4 the positive system (1) is reachable at the segment of line $\{[1,0],[1,2]\}$ since $A \in M_3$ is diagonal and B is monomial. It is easy to check that the conditions (17) are met and the minimum energy control problem has a solution.

Using the Procedure 1 we obtain the following:
Step 1. In this case we have

$$
e^{At} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \tag{34}
$$

Step 2. Using (15), (28) we obtain

$$
\begin{aligned}
W &= \int_0^{t_f} e^{A(t_f-\tau)} \bar{B} Q^{-1} \bar{B}^T e^{A^T(t_f-\tau)} d\tau = \int_0^{t_f} \begin{bmatrix} 1.5e^{-2\tau} & 0 & 0 \\ 0 & 1.5e^{4\tau} & 0 \\ 0 & 0 & 1.5e^{-2\tau} \end{bmatrix} d\tau \\
&= \begin{bmatrix} \frac{3}{4}(1-e^{-2t_f}) & 0 & 0 \\ 0 & \frac{3}{8}(e^{4t_f}-1) & 0 \\ 0 & 0 & \frac{3}{4}(1-e^{-2t_f}) \end{bmatrix}.
\end{aligned} \tag{35}
$$

Step 3. Using (16) and (34) we obtain

$$\hat{u}(t) = Q^{-1}\bar{B}^T e^{A^T(t_f-t)}W^{-1}x_f = \text{diag}[0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5]\times$$

$$\times \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-(t_f-t)} & 0 & 0 \\ 0 & e^{2(t_f-t)} & 0 \\ 0 & 0 & e^{-(t_f-t)} \end{bmatrix} \times$$

(36)

$$\times \begin{bmatrix} \frac{3}{4}(1-e^{-2t_f}) & 0 & 0 \\ 0 & \frac{3}{8}(e^{4t_f}-1) & 0 \\ 0 & 0 & \frac{3}{4}(1-e^{-2t_f}) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{4e^{2(t_f-t)}}{3(e^{4t_f}-1)} & \frac{2e^{-(t_f-t)}}{3(1-e^{-2t_f})} & \frac{2e^{-(t_f-t)}}{3(1-e^{-2t_f})} & \frac{4e^{2(t_f-t)}}{3(e^{4t_f}-1)} & \frac{2e^{-(t_f-t)}}{3(1-e^{-2t_f})} \\ & & \frac{2e^{-(t_f-t)}}{3(1-e^{-2t_f})} & \frac{4e^{2(t_f-t)}}{3(e^{4t_f}-1)} & \frac{2e^{-(t_f-t)}}{3(1-e^{-2t_f})} & \frac{2e^{-(t_f-t)}}{3(1-e^{-2t_f})} \end{bmatrix}^T.$$

Step 4. Using (18) and (35) we obtain

$$I(\hat{u}) = x_f^T W^{-1} x_f = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \times$$

$$\times \begin{bmatrix} \frac{3}{4}(1-e^{-2}) & 0 & 0 \\ 0 & \frac{3}{8}(e^4-1) & 0 \\ 0 & 0 & \frac{3}{4}(1-e^{-2}) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{3}{8(e^4-1)} + \frac{8}{3(1-e^{-2})}$$

(37)

From Fig. 1 it follows that the condition (14) for (32) is satisfied for $t_f = 1$, $U < 1$.

5. Concluding remarks

A new formulation and solution to the minimum energy control problem for positive 2D continuous-discrete linear systems with bounded inputs have been proposed. New necessary and sufficient conditions for the positivity of a class of 2D continuous-discrete linear systems have been established (Theorem 1). A new notion of the reachability to the segment of line has been introduced and necessary and sufficient conditions for the reachability have been presented (Theorem 4). Conditions for the existence of the solution to the minimum energy control problem (Theorem 5) and procedures for
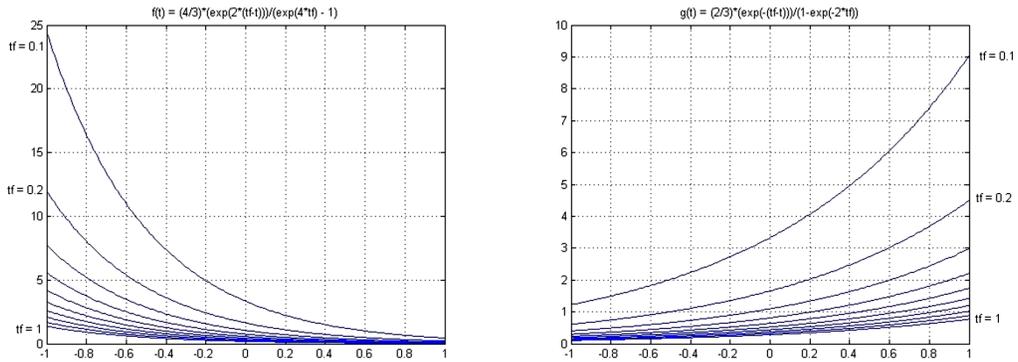
Figure 1: Input vector

computation of an input minimizing the given performance index have been proposed. Effectiveness of the procedure have been demonstrated on numerical example.

An open problem is an extension of the minimum energy control problem to standard and positive 2D continuous-discrete linear systems described by the equation [15, 21]

$$\dot{x}(t,i+1) = A_0 x(t,i) + A_1 \dot{x}(t,i) + A_2 x(t,i+1) + B_0 u(t,i) + B_1 \dot{u}(t,i) + B_2 u(t,i+1)$$

where $\dot{x}(t,i) = \frac{\partial x(t,i)}{\partial t}$, $x(t,i) \in \Re^n$, $u(t,i) \in \Re^m$ are the state and input vectors and $A_k \in \Re^{n\times n}$, $B_k \in \Re^{n\times m}$, $k = 0, 1, 2$.

References

[1] Y. BISTRITZ: A stability test for continuous-discrete bivariate polynomials. *Proc. Int. Symp. on Circuits and Systems*, **3** (2003), 682-685.

[2] M. BUSŁOWICZ: Stability and robust stability conditions for a general model of scalar continuous-discrete linear systems. *Measurement Automation and Monitoring*, **56**(2), (2010), 133-135.

[3] BUSŁOWICZ: Robust stability of the new general 2D model of a class of continuous-discrete linear systems. *Bulletin of the Polish Academy of Sciences Technical Sciences*, **58**(4), (2010), 561-565.

[4] M. DYMKOV, I. GAISHUN, E. ROGERS, K. GAŁKOWSKI and D.H. OWENS: Control theory for a class of 2D continuous-discrete linear systems. *Int. J. Control*, **77**(9), (2004), 847-860.

[5] L. FARINA and S. RINALDI: Positive Linear Systems; Theory and Applications. J. Wiley, New York, 2000.

[6] K. GAŁKOWSKI, E. ROGERS, W. PASZKE and D.H. OWENS: Linear repetitive process control theory applied to a physical example. *Int. J. of Applied Mathematics and Computer Science*, **13**(1), (2003), 87-99.

[7] T. KACZOREK: Minimum energy control of fractional positive continuous-time linear systems. *Proc. of Int. Conf. on Methods and Models in Automation and Robotics*, Międzyzdroje, Poland, (2013).

[8] T. KACZOREK: Minimum energy control of descriptor positive discrete-time linear systems. *COMPEL – The Int. J. for Computation and Mathematics in Electrical and Electronic Engineering*, **33**(3), (2014), 976-988.

[9] T. KACZOREK: Minimum energy control of positive continuous-time linear systems. *Int. J. of Applied Mathematics and Computer Science*, (in Press).

[10] T. KACZOREK: Minimum energy control of positive continuous-time linear systems with bounded inputs. *Int. J. of Applied Mathematics and Computer Science*, (in Press).

[11] T. KACZOREK: Minimum energy control of positive discrete-time linear systems with bounded inputs. *Proc. Int. Conf. on Mathematical Methods in Engineering*, Porto, Portugal, (2013).

[12] T. KACZOREK: Minimum energy control of positive fractional continuous-time linear systems. *Bulletin of the Polish Academy of Sciences Technical Sciences*, (in Press).

[13] T. KACZOREK: Minimum energy control of positive fractional descriptor continuous-time linear systems. *IET Control Theory & Applications*, **8**(4), (2014), 219-225.

[14] T. KACZOREK: New stability tests of positive standard and fractional linear systems. *Circuit and Systems*, **2**(4), (2011), 261-268.

[15] T. KACZOREK: Positive 1D and 2D systems. Springer Verlag, London, 2001.

[16] T. KACZOREK: Positive 2D hybrid linear systems. *Bulletin of the Polish Academy of Sciences Technical Sciences*, **55**(4), (2007), 351-358.

[17] T. KACZOREK: Positive fractional 2D continuous-discrete linear systems. *Bulletin of the Polish Academy of Sciences Technical Sciences*, **59**(4), (2011), 575-579.

[18] T. KACZOREK: Positive fractional 2D hybrid linear systems. *Bulletin of the Polish Academy of Sciences Technical Sciences*, **56**(3), (2008), 273-277.

[19] T. KACZOREK: Reachability and minimum energy control of positive 2D continuous-discrete systems. *Bulletin of the Polish Academy of Sciences Technical Sciences*, **46**(1), (1998), 85-93.

[20] T. KACZOREK: Realization problem for positive 2D hybrid systems. *COMPEL – The Int. J. for Computation and Mathematics in Electrical and Electronic Engineering*, **27**(3), (2008), 613-623.

[21] T. KACZOREK: Selected Problems of Fractional Systems Theory. Springer-Verlag, Berlin, 2012.

[22] T. KACZOREK: Stability of continuous-discrete linear systems described by general model. *Bulletin of the Polish Academy of Sciences Technical Sciences*, **59**(2), (2011), 189-193.

[23] T. KACZOREK and J. KLAMKA: Minimum energy control of 2D linear systems with variable coefficients. *Int. J. of Control*, **44**(3), (1986), 645-650.

[24] J. KLAMKA: Controllability and minimum energy control problem of fractional discrete-time systems. Chapter in "New Trends in Nanotechology and Fractional Calculus", Eds. Baleanu D., Guvenc Z.B., Tenreiro Machado J.A., Springer-Verlag, New York, 2010, 503-509.

[25] J. KLAMKA: Controllability of Dynamical Systems. Kluwer Academic Press, Dordrecht, 1991.

[26] J. KLAMKA: Controllability of dynamical systems-a survey. *Archives of Control Sciences*, **2**(3-4), (1993), 281-307.

[27] J. KLAMKA: Minimum energy control of 2D systems in Hilbert spaces. *Systems Science*, **9**(1-2), (1983), 33-42.

[28] J. KLAMKA: Minimum energy control of discrete systems with delays in control. *Int. J. of Control*, **26**(5), (1977), 737-744.

[29] J. KLAMKA: Relative controllability and minimum energy control of linear systems with distributed delays in control. *IEEE Trans. on Automatic Control*, **21**(4), (1976), 594-595.

[30] T. KACZOREK, V. MARCHENKO and L. SAJEWSKI: Solvability of 2D hybrid linear systems - comparison of three different methods. *Acta Mechanica et Automatica*, **2**(2), (2008), 59-66.

[31] K.S. NARENDRA and R. SHORTEN: Hurwitz stability of Metzler matrices. *IEEE Trans. on Automatic Control*, **55**(6), (2010), 1484-1487.

[32] L. SAJEWSKI: Solution of 2D singular hybrid linear systems. *Kybernetes*, **38**(7-8), (2009), 1079-1092.

[33] Y. XIAO: Stability test for 2-D continuous-discrete systems. *Proc. 40th IEEE Conf. on Decision and Control*, **4** (2001), 3649-3654.

[34] Y. XIAO: Stability, controllability and observability of 2-D continuous-discrete systems. *Proc. Int. Symp. on Circuits and Systems*, **4** (2003), 468-471.