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Lyapunov matrices approach to the parametric optimization of a neutral system

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In the paper a Lyapunov matrices approach to the parametric optimization problem of a neutral system with a P-controller is presented. The value of integral quadratic performance index of quality is equal to the value of Lyapunov functional for the initial function of the neutral system. The Lyapunov functional is determined by means of the Lyapunov matrix.

Key words: neutral system, Lyapunov matrix, Lyapunov functional.

1. Introduction

The Lyapunov functionals are used to test the stability of systems, in calculation of the robustness bounds for uncertain time delay systems, in computation of the exponential estimates for the solutions of time delay systems. The method of determination of a Lyapunov functional for a time delay system with one delay was presented by Repin [15]. Duda [3] used the Repin's method to determination of a Lyapunov functional for a system with two delays, for a system with both lumped and distributed delay [5,10], for a system with time-varying delay [6,8]. In last years a method of determination of a Lyapunov functional by means of Lyapunov matrices is very popular, see for example [9,12,13,14,16].

The Lyapunov quadratic functionals are also used to calculation of a value of a quadratic performance index of quality in the process of the parametric optimization for time delay systems, see for instance [2,4,7,9]. One constructs a functional for a system with a time delay with a given time derivative whose is equal to the negatively defined quadratic form of a system state. The value of that functional at the initial state of a time delay system is equal to the value of a quadratic performance index of quality. In the paper a Lyapunov matrices approach to the parametric optimization problem of a neutral system is presented. This paper extends the results of Duda [9] to a neutral system.

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2. Preliminaries

Let us consider a neutral system

$$\begin{cases} \frac{dx(t)}{dt} - C\frac{dx(t-r)}{dt} = Ax(t) + Bx(t-r) \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{1}$$

for $t \geqslant t_0$, $\theta \in [-r,0]$, $r > 0$, where $x(t) \in \mathbb{R}^n$, A, B, $C \in \mathbb{R}^{n\times n}$, function $\varphi \in PC([-r,0],\mathbb{R}^n)$ - the space of piece-wise continuous vector valued functions defined on the segment $[-r,0]$ with the uniform norm $\| \varphi \|_{PC} = \sup_{\theta\in[-r,0]} \| \varphi(\theta) \|$.

Definition 3 *The* ***difference equation*** *associated with* (1) *is given by a term*

$$x(t) - Cx(t-r) = 0 \quad for\, t \geqslant t_0 \tag{2}$$

We assume that the difference $x(t) - Cx(t-r)$ is continuous and differentiable for $t \geqslant t_0$, except possibly a countable number of points.

Definition 4 *The* ***spectrum*** $\sigma(C)$ *is a set of complex numbers* λ *for which a matrix* $\lambda I - C$ *is not invertible.*

$$\sigma(C) = \{\lambda \in \mathbb{C} : \det(\lambda I - C) = 0\} \tag{3}$$

The arbitrary eigenvalue of the matrix C *will be denoted as* $\lambda(C)$.

Definition 5 *The* ***spectral radius*** *of a matrix* C *is given by a form*

$$\gamma(C) = \sup\{|\lambda| : \lambda \in \sigma(C)\} \tag{4}$$

Definition 6 *The matrix* C *is called a* ***Schur stable matrix*** *if the eigenvalues of* C *lie in the interior of the unit disk of the complex plane, i.e. if the spectral radius* $\gamma(C) < 1$.

Let $x(t,\varphi)$ be the solution of system (1) with the initial function φ for $t \geqslant t_0$.

Definition 7 *The function* $x_t(\varphi) : [-r,0] \to \mathbb{R}^n$ *is called a* ***shifted restriction*** *of* $x(\cdot,\varphi)$ *to an interval* $[t-r,t]$ *and is defined by a formula*

$$x_t(\varphi)(\theta) := x(t+\theta,\varphi) \tag{5}$$

for $t \geqslant t_0$ *and* $\theta \in [-r,0]$.

The initial condition for equation (1) can be written in a form

$$x_{t_0}(\varphi) = \varphi \tag{6}$$

Definition 8 (Bellman & Cooke [1]). *The trivial solution of* (1) *is said to be* ***stable*** *if for any* $\varepsilon > 0$ *there is* $\delta > 0$ *such that*

$$\| \varphi \|_{PC} \leqslant \delta \Rightarrow \| x(t,\varphi) \| \leqslant \varepsilon$$

for every $t \geqslant t_0$.

Definition 9 (Bellman & Cooke [1]). *The trivial solution of* (1) *is said to be* ***asymptotically stable*** *if it is stable and* $x(t,\varphi) \to 0$ *as* $t \to \infty$

Definition 10 (Bellman & Cooke [1]). *The trivial solution of* (1) *is said to be* ***exponentially stable*** *if there exist* $M \geqslant 1$ *and* $\sigma > 0$ *such that for every solution* $x(t,\varphi)$ *of the system with initial function* $\varphi \in PC([-r,0],\mathbb{R}^n)$*, the following condition holds*

$$\| x(t,\varphi) \| \leqslant M \| \varphi \|_{PC} e^{-\sigma t}$$

for every $t \geqslant t_0$.

The eigenvalues of the neutral equation (1) for large modulus are asymptotically equal to the eigenvalues of the difference equation (2).

According to the Theorem 9.6.1 (Hale & Verduyn Lunel [11]) the difference equation (2) is stable when the matrix C is Schur stable.

When the matrix C is Schur stable, then the asymptotic stability of system (1) is equivalent to the exponential stability of the system (1). We assume that C is not singular and a Schur stable matrix.

Definition 11 (Bellman & Cooke [1]). $K(t)$ *is the* ***fundamental matrix*** *of system* (1) *if it satisfies the matrix equation*

$$\frac{d}{dt}K(t) - C\frac{d}{dt}K(t-r) = AK(t) + BK(t-r)$$

for $t \geqslant 0$ *and the following conditions*

- *initial condition*: $K(0) = I_{n\times n}$ and $K(t) = 0_{n\times n}$ for $t < 0$ where $I_{n\times n}$ is the identity $n\times n$ matrix and $0_{n\times n}$ is the zero $n\times n$ matrix,
- *sewing condition*: $K(t) - CK(t-r)$ is continuous for $t > 0$.

It follows from the definition that the fundamental matrix $K(t)$ has discontinuity points.

The sewing condition implies *the jump equation*

$$\triangle K(t) - C\triangle K(t-r) = 0 \tag{7}$$

for $t \geqslant 0$, where $\triangle K(t) = K(t+0) - K(t-0)$

To compute the size of the jumps one needs to solve the jump equation (7) at $t_j = jr$, $j = 0,1,2,..$, with the initial condition $\triangle K(0) = I$

Lemma 1 (Bellman & Cooke [1]). *The fundamental matrix* $K(t)$ *has jumps at points* $t_j = jr,\ j = 0,1,2,\ldots$

$$\triangle K(t)\mid_{t=t_j} = K(jr+0) - K(jr-0) = C^j \tag{8}$$

and $K(t) = K(t+0)$ *at the jump points.*

Theorem 14 (Bellman & Cooke [1]). *Let* $K(t)$ *be the fundamental matrix of system* (1), *then for* $t \geqslant t_0$

$$x(t,\varphi) = [K(t-t_0) - K(t-t_0-r)]\,\varphi(0) + \int_{-r}^{0} K(t-t_0-r-\theta)\left[B\varphi(\theta) + C\frac{d}{d\theta}\varphi(\theta)\right]d\theta \tag{9}$$

This expression is called *the Cauchy formula* for system (1).

Theorem 15 (Bellman & Cooke [1]). *The fundamental matrix* $K(t)$ *of system* (1) *satisfies also the equation*

$$\frac{d}{dt}K(t) - \frac{d}{dt}K(t-r)C = K(t)A + K(t-r)B \tag{10}$$

for $t > 0$ *and* $t \neq jr,\ j = 1,2,\ldots$

3. A Lyapunov-Krasovskii functional

Given a symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$. We are looking for a functional $v : PC([-r,0],\mathbb{R}^n) \to \mathbb{R}$ such that along the solutions of system (1) the following equality holds

$$\frac{d}{dt}v(x_t(\varphi)) = -x^T(t,\varphi)Wx(t,\varphi) \tag{11}$$

for $t \geqslant t_0$, where $x(t,\varphi)$ is a solution of system (1), with the initial function $\varphi \in PC([-r,0],\mathbb{R}^n)$, given by (9) and $x_t(\varphi)$ is a shifted restriction of $x(\cdot,\varphi)$ to an interval $[t-r,t]$, given by (5).

We assume that system (1) is asymptotically stable and integrate both sides of Eq. (11) from t_0 to infinity. We obtain

$$v(x_{t_0}(\varphi)) = \int_{t_0}^{\infty} x^T(t,\varphi)Wx(t,\varphi)dt \tag{12}$$

Taking into account (9) we calculate the integral of the right-hand side of Eq. (12)

$$\int_{t_0}^{\infty} x^T(t,t_0,\varphi)Wx(t,t_0,\varphi)dt =$$

$$= \varphi^T(0)\int_0^{\infty} K^T(t)WK(t)dt\varphi(0) - \varphi^T(0)\int_0^{\infty} K^T(t)WK(t-r)dtC\varphi(0) +$$

$$-\varphi^T(0)C^T\int_0^{\infty} K^T(t-r)WK(t)dt\varphi(0) + \varphi^T(0)C^T\int_0^{\infty} K^T(t-r)WK(t-r)dtC\varphi(0) +$$

$$+2\varphi^T(0)\int_{-r}^{0}[\int_0^{\infty} K^T(t)WK(t-r-\theta)dt][B\varphi(\theta)+C\frac{d}{d\theta}\varphi(\theta)]d\theta +$$

$$-2\varphi^T(0)C^T\int_{-r}^{0}[\int_0^{\infty} K^T(t-r)WK(t-r-\theta)dt][B\varphi(\theta)+C\frac{d}{d\theta}\varphi(\theta)]d\theta +$$

$$+\int_{-r}^{0}\int_{-r}^{0}[\varphi^T(\theta)B^T + \frac{d}{d\theta}\varphi^T(\theta)C^T]\cdot$$

$$\cdot[\int_0^{\infty} K^T(t-r-\theta)WK(t-r-\xi)dt][B\varphi(\xi)+C\frac{d}{d\xi}\varphi(\xi)]d\theta d\xi \tag{13}$$

Definition 12 *We introduce **a Lyapunov matrix***

$$U(\xi) = \int_0^{\infty} K^T(t)WK(t+\xi)dt \tag{14}$$

Lemma 2 (Rodriguez et al. [16]). *Let system* (1) *be exponentially stable. Then for every symmetric matrix $W \in \mathbb{R}^{n\times n}$, matrix $U(\xi)$ is well defined and satisfies the following properties:*

Dynamic property

$$\frac{d}{d\xi}U(\xi) - \frac{d}{d\xi}U(\xi-r)C = U(\xi)A + U(\xi-r)B \tag{15}$$

for $\xi \geqslant 0$ and $\xi \neq jr,\ j = 0,1,2,\ldots$

Symmetry property

$$U(-\xi) = U^T(\xi) \tag{16}$$

for $\xi \geqslant 0$

Algebraic property

$$-W = A^T U(0) + U(0)A - A^T U(-r)C - C^T U^T(-r)A + B^T U^T(-r) + \\ +U(-r)B - B^T U(0)C - C^T U(0)B \tag{17}$$

Using the Lyapunov matrix (14) we attain a formula for the functional $v(x_{t_0}(\varphi))$

$$v(x_{t_0}(\varphi)) = \varphi^T(0)[U(0) - U(-r)C - C^T U^T(-r) + C^T U(0)C]\varphi(0) +$$

$$+2\varphi^T(0)\int_{-r}^{0}[U(-\theta - r) - C^T U(-\theta)][B\varphi(\theta) + C\frac{d}{d\theta}\varphi(\theta)]d\theta +$$

$$+\int_{-r}^{0}\int_{-r}^{0}[B\varphi(\theta) + C\frac{d}{d\theta}\varphi(\theta)]^T U(\theta - \xi)[B\varphi(\xi) + C\frac{d}{d\xi}\varphi(\xi)]d\theta d\xi \tag{18}$$

Using the symmetry property one can express the formula (18) in a form

$$v(x_{t_0}(\varphi)) = \varphi^T(0)[U(0) - U^T(r)C - C^T U(r) + C^T U(0)C]\varphi(0) +$$

$$+2\varphi^T(0)\int_{-r}^{0}[U(\theta + r) - U(\theta)C]^T[B\varphi(\theta) + C\frac{d}{d\theta}\varphi(\theta)]d\theta +$$

$$+\int_{-r}^{0}\int_{-r}^{0}[B\varphi(\theta) + C\frac{d}{d\theta}\varphi(\theta)]^T U(\theta - \xi)[B\varphi(\xi) + C\frac{d}{d\xi}\varphi(\xi)]d\theta d\xi \tag{19}$$

Using Eq. (6) one can express a relation (19) more general in a form

$$v(x_{t_0}(\varphi)) = x_{t_0}(\varphi)^T(0)[U(0) - U^T(r)C - C^T U(r) + C^T U(0)C]x_{t_0}(\varphi)(0) +$$

$$+2x_{t_0}^T(\varphi)(0)\int_{-r}^{0}[U(\theta + r) - U(\theta)C]^T[Bx_{t_0}(\varphi)(\theta) + C\frac{d}{d\theta}x_{t_0}(\varphi)(\theta)]d\theta +$$

$$+\int_{-r}^{0}\int_{-r}^{0}[Bx_{t_0}(\varphi)(\theta)+C\frac{d}{d\theta}x_{t_0}(\varphi)(\theta)]^T U(\theta-\xi)[Bx_{t_0}(\varphi)(\xi)+C\frac{d}{d\xi}x_{t_0}(\varphi)(\xi)]d\theta d\xi \quad (20)$$

Lemma 3 (Rodriguez et al. [16]) *The Lyapunov matrix $U(\xi)$ for system* (1) *is continuously differentiable at $\xi \neq jr$, $j = 0,1,2,\ldots$, and at $\xi = jr$ matrix $\frac{d}{d\xi}U(\xi)$ has the jump*

$$\frac{d}{d\xi}U(jr+0)-\frac{d}{d\xi}U(jr-0)=-(Q-W)C^j \quad (21)$$

Here Q is the solution of the matrix equation

$$Q-C^TQC=W \quad (22)$$

4. A Lyapunov matrix for a neutral system

To obtain a Lyapunov matrix for a neutral system one needs to solve a set of equations (see Lemma 2)

$$\frac{d}{d\xi}U(\xi)-\frac{d}{d\xi}U(\xi-r)C=U(\xi)A+U(\xi-r)B \quad (23)$$

$$U(-\xi)=U^T(\xi) \quad (24)$$

$$\begin{aligned}-W=&A^TU(0)+U(0)A-A^TU(-r)C-C^TU^T(-r)A+\\&+B^TU^T(-r)+U(-r)B-B^TU(0)C-C^TU(0)B\end{aligned} \quad (25)$$

Formula (24) implies

$$U(\xi-r)=U^T(-\xi+r) \quad (26)$$

and Eq. (23) takes a form

$$\frac{d}{d\xi}U(\xi)-\frac{d}{d\xi}U^T(-\xi+r)C=U(\xi)A+U^T(-\xi+r)B \quad (27)$$

We introduce a new variable $\tau=-\xi+r$. The term (27) for a new variable has a form

$$\frac{d}{d\tau}U^T(-\tau+r)-C^T\frac{d}{d\tau}U(\tau)=-A^TU^T(-\tau+r)-B^TU(\tau) \quad (28)$$

One obtains a set of equations

$$\begin{cases} \frac{d}{d\xi}U(\xi) - \frac{d}{d\xi}U^T(-\xi+r)C = U(\xi)A + U^T(-\xi+r)B \\ \frac{d}{d\xi}U^T(-\xi+r) - C^T\frac{d}{d\xi}U(\xi) = -A^TU^T(-\xi+r) - B^TU(\xi) \end{cases} \tag{29}$$

We introduce a new function

$$Z(\xi) = U^T(-\xi+r) \tag{30}$$

The set of equations (29) can be written in a form

$$\begin{cases} \frac{d}{d\xi}U(\xi) - \frac{d}{d\xi}Z(\xi)C = U(\xi)A + Z(\xi)B \\ \frac{d}{d\xi}Z(\xi) - C^T\frac{d}{d\xi}U(\xi) = -A^TZ(\xi) - B^TU(\xi) \end{cases} \tag{31}$$

or in a equivalent form

$$\begin{cases} \frac{d}{d\xi}U(\xi) - C^T\frac{d}{d\xi}U(\xi)C = U(\xi)A - B^TU(\xi)C + Z(\xi)B - A^TZ(\xi)C \\ \frac{d}{d\xi}Z(\xi) - C^T\frac{d}{d\xi}Z(\xi)C = -B^TU(\xi) + C^TU(\xi)A - A^TZ(\xi) + C^TZ(\xi)B \end{cases} \tag{32}$$

for $\xi \in [0,r]$ with the initial conditions $U(0)$ and $Z(0)$.

The formulas (24) and (30) imply

$$U(-r) = U^T(r) = Z(0) \tag{33}$$

Taking into account (33) one can write the algebraic property (25) in a form

$$\begin{aligned} -W &= A^TU(0) + U(0)A - A^TZ(0)C - C^TZ^T(0)A + \\ &\qquad + B^TZ^T(0) + Z(0)B - B^TU(0)C - C^TU(0)B \end{aligned} \tag{34}$$

Solution of Eq. (32) is given

$$\begin{bmatrix} U(\xi) \\ Z(\xi) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(\xi) & \Phi_{12}(\xi) \\ \Phi_{21}(\xi) & \Phi_{22}(\xi) \end{bmatrix} \begin{bmatrix} U(0) \\ Z(0) \end{bmatrix} \tag{35}$$

where a matrix $\Phi(\xi) = \begin{bmatrix} \Phi_{11}(\xi) & \Phi_{12}(\xi) \\ \Phi_{21}(\xi) & \Phi_{22}(\xi) \end{bmatrix}$ is a fundamental matrix of system (32).

We determine the initial conditions $U(0)$, $Z(0)$. The term (30) implies $U(r) = Z^T(0)$ and $Z(r) = U^T(0) = U(0)$.

From Eq. (35) we obtain

$$U(r) = Z^T(0) = \Phi_{11}(r)U(0) + \Phi_{12}(r)Z(0) \tag{36}$$

$$Z(r) = U(0) = \Phi_{21}(r)U(0) + \Phi_{22}(r)Z(0) \tag{37}$$

We put (36) into (34) and reshape (37). In this way we attain a set of algebraic equations which enables us to calculate the initial conditions of Eq. (35).

$$[A^T + B^T\Phi_{11}(r)]U(0) + U(0)A - C^T\Phi_{11}(r)U(0)A - B^T U(0)C - C^T U(0)B+$$

$$+B^T\Phi_{12}(r)Z(0) + Z(0)B - A^T Z(0)C - C^T\Phi_{12}(r)Z(0)A = -W \tag{38}$$

$$[\Phi_{21}(r) - I]U(0) + \Phi_{22}(r)Z(0) = 0 \tag{39}$$

5. Formulation of the parametric optimization problem

Let us consider a neutral system with a P-controller

$$\begin{cases} \frac{dx(t)}{dt} - C\frac{dx(t-r)}{dt} = Ax(t) + Bu(t-r) \\ u(t) = -Px(t) \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{40}$$

for $t \geqslant t_0$, $\theta \in [-r,0]$, where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $A, C \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $P \in \mathbb{R}^{p\times n}$ is a P-controller gain, $\varphi \in PC([-r,0],\mathbb{R}^n)$

System (40) can be written in an equivalent form

$$\begin{cases} \frac{dx(t)}{dt} - C\frac{dx(t-r)}{dt} = Ax(t) - BPx(t-r) \\ x(t_0+\theta) = \varphi(\theta) \end{cases} \tag{41}$$

In parametric optimization problem will be used the performance index of quality

$$J = \int_{t_0}^{\infty} x^T(t,\varphi)Wx(t,\varphi)dt \tag{42}$$

where $W \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and $x(t,\varphi)$ is a solution of Eq. (41) for initial function φ.

Problem Determine the matrix $P \in \mathbb{R}^{p\times n}$ whose minimize an integral quadratic performance index of quality (42).

According to Eq. (12) the value of the performance index of quality (42) is equal to the value of the functional (20) for initial function φ. To calculate the value of the functional (20) we need a Lyapunov matrix $U(\xi)$. To obtain a Lyapunov matrix $U(\xi)$ we have to solve a system of Eqs. (15), (16), (17).

6. Example

Let us consider a neutral system with a P-controller

$$\begin{cases} \frac{dx(t)}{dt} - c\frac{dx(t-r)}{dt} = ax(t) + bu(t-r) \\ u(t) = -px(t) \\ x(0) = x_0 \\ x(\theta) = 0 \end{cases} \tag{43}$$

$t \geqslant 0$, $x(t), u(t) \in \mathbb{R}$, $\theta \in [-r, 0]$, $r \geqslant 0$. The parameter p is a gain of a P-controller, $x_0 \in \mathbb{R}$ is an initial state of a system.

One can reshape Eq. (43) to a form

$$\begin{cases} \frac{dx(t)}{dt} - c\frac{dx(t-r)}{dt} = ax(t) - bpx(t-r) \\ x(0) = x_0 \\ x(\theta) = 0 \end{cases} \tag{44}$$

for $t \geqslant 0$ and $\theta \in [-r, 0)$. The initial function φ is given by a term

$$\varphi(\theta) = \begin{cases} x_0 & for\; \theta = 0 \\ 0 & for\; \theta \in [-r, 0) \end{cases} \tag{45}$$

In parametric optimization problem we use the performance index

$$J = \int_0^\infty wx^2(t, \varphi)dt \tag{46}$$

where $w > 0$ and $x(t, \varphi)$ is a solution of (44) for initial function (45).

System of equations (32) takes a form

$$\begin{bmatrix} \frac{d}{d\xi}U(\xi) \\ \frac{d}{d\xi}Z(\xi) \end{bmatrix} = \begin{bmatrix} \frac{a+bcp}{1-c^2} & -\frac{ac+bp}{1-c^2} \\ \frac{ac+bp}{1-c^2} & -\frac{a+bcp}{1-c^2} \end{bmatrix} \begin{bmatrix} U(\xi) \\ Z(\xi) \end{bmatrix} \tag{47}$$

A fundamental matrix of solutions of Eq. (47) has a form

$$\Phi(\xi) = \begin{bmatrix} \cosh\lambda\xi + \frac{a+bcp}{\lambda(1-c^2)}\sinh\lambda\xi & -\frac{ac+bp}{\lambda(1-c^2)}\sinh\lambda\xi \\ \frac{ac+bp}{\lambda(1-c^2)}\sinh\lambda\xi & \cosh\lambda\xi - \frac{a+bcp}{\lambda(1-c^2)}\sinh\lambda\xi \end{bmatrix} \tag{48}$$

where

$$\lambda = \sqrt{\frac{a^2 - b^2p^2}{1 - c^2}} \tag{49}$$

Initial conditions of system (47) one obtains solving of the algebraic equation

$$\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} U(0) \\ Z(0) \end{bmatrix} = \begin{bmatrix} -w \\ 0 \end{bmatrix} \tag{50}$$

where

$q_{11} = 2a + 2bcp - (ac+bp)(\cosh\lambda r + \frac{a+bcp}{\lambda(1-c^2)}\sinh\lambda r)$

$q_{12} = (ac+bp)(\frac{ac+bp}{\lambda(1-c^2)}\sinh\lambda r - 1)$

$q_{21} = \frac{ac+bp}{\lambda(1-c^2)}\sinh\lambda r - 1$

$q_{22} = \cosh\lambda r - \frac{a+bcp}{\lambda(1-c^2)}\sinh\lambda r$

Solving Eq. (50) we obtain

$$U(0) = \frac{w}{M}\left[-\cosh\lambda r + \frac{a+bcp}{\lambda(1-c^2)}\sinh\lambda r\right] \tag{51}$$

$$Z(0) = \frac{w}{M}\left[\frac{ac+bp}{\lambda(1-c^2)}\sinh\lambda r - 1\right] \tag{52}$$

where

$$M = 2(a+bcp)\cosh\lambda r - 2\lambda(1-c^2)\sinh\lambda r - 2(ac+bp) \tag{53}$$

Solution of Eq. (47) has a form

$$U(\xi) = \frac{w}{M}\left[-\cosh\lambda r + \frac{a+bcp}{\lambda(1-c^2)}\sinh\lambda r\right]\cosh\lambda\xi - \frac{w}{2\lambda(1-c^2)}\sinh\lambda\xi \tag{54}$$

$$\begin{aligned} Z(\xi) = \frac{w}{M}\left[\frac{ac+bp}{\lambda(1-c^2)}\sinh\lambda r - 1\right]\cosh\lambda\xi + \\ + \frac{w}{M\lambda(1-c^2)}\left[a+bcp-(ac+bp)\cosh\lambda r\right]\sinh\lambda\xi \end{aligned} \tag{55}$$

The value of the performance index (46) is equal to the value of the functional (20) for initial function. In this example initial function is given by (45). After calculations one obtains

$$J = \frac{\frac{wx_0^2}{2}(2c - (1+c^2)\cosh\lambda r + \frac{a-bcp}{\lambda}\sinh\lambda r)}{-ac - bp + (a+bcp)\cosh\lambda r - \lambda(1-c^2)\sinh\lambda r} \tag{56}$$

We search for an optimal gain which minimize the index (56). Optimization results are given in Table 1. These results are obtained for $x_0 = 1$, $w = 1$, $a = -1$, $b = 0.5$, and $c = -0.6$. Critical gain is a maximal admissible gain for system (43). System (43) is unstable for gains greater then critical gain.

Table 1. Result of the optimization

Delay r	*Optimal gain*	*Index value*	*Critical gain*
0.5	3.2	0.41	8
1	1.4	0.52	4.5
1.5	0.8	0.55	3.4
2	0.6	0.56	3
2.5	0.45	0.56	2.7

7. Conclusions

In the paper a Lyapunov matrices approach to the parametric optimization problem of neutral system is presented. The value of integral quadratic performance index of quality is equal to the value of Lyapunov functional for the initial function of neutral system. The Lyapunov functional is determined by means of the Lyapunov matrix.

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