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Drazin inverse matrix method for fractional descriptor discrete-time linear systems

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Abstract. The Drazin inverse of matrices is applied in order to find the solutions of the state equations of fractional descriptor discrete-time linear systems. The solution of the state equation is derived and the set of consistent initial conditions for a given set of admissible inputs is established. The proposed method is illustrated by a numerical example.

Key words: Drazin inverse, descriptor, fractional, discrete-time, linear system solution.

3. Introduction

Descriptor (singular) linear systems have been considered in many papers and books [1–19]. The eigenvalues and invariants assignment by state and output feedbacks have been investigated in [8, 12, 17], and the minimum energy control of descriptor linear systems in [20, 21]. The computation of Kronecker's canonical form of singular pencil has been analyzed in [19]. The positive linear systems with different fractional orders have been addressed in [22]. Selected problems in theory of fractional linear systems have been described in monograph [16].

Descriptor and standard positive linear systems with the use of Drazin inverse have been addressed in [1–4, 10, 13, 17]. The shuffle algorithm has been applied to checking the positivity of descriptor linear systems in [11]. The stability of positive descriptor systems has been investigated in [23]. Reduction and decomposition of descriptor fractional discrete-time linear systems have been considered in [14]. A new class of descriptor fractional linear discrete-time system has been introduced in [15].

The Drazin inverse for finding the solution to the state equation of fractional continuous-time linear systems has been applied in [10] and the controllability, reachability and minimum energy control of fractional discrete-time linear systems with delays in state have been investigated in [24]. A comparison of three different methods for finding the solution for descriptor fractional discrete-time linear system has been presented in [25].

In this paper the solution to the state equation of fractional descriptor discrete-time linear systems by the use of Drazin inverse of matrices will be derived.

The paper is organized as follows. In section 2 the state equation of the fractional descriptor discrete-time linear systems and some basic definitions of the Drazin inverse and its properties are recalled. The solution to the state equation is presented in section 3 and illustrated with a numerical example in section 4. Concluding remarks are given in section 5.

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The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices and $\Re^n=\Re^{n\times 1}$, Z_+ – the set of nonnegative integers, I_n – the $n\times n$ identity matrix, ker A (im A) – the kernel (image) of the matrix.

3. Fractional descriptor discrete-time linear systems

Consider the fractional descriptor discrete-time linear system

$$E\Delta^\alpha x_{i+1}=Ax_i+Bu_i,\ i\in Z_+=\{0,1,\dots\},\tag{1}$$

where $x_i\in\Re^n$ is the state vector $u_i\in\Re^m$ is the input vector, $E,A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$ and

$$\Delta^\alpha x_i=\sum_{j=0}^{i}(-1)^j\binom{\alpha}{j}x_{i-j}\tag{2a}$$

$$\binom{\alpha}{j}=\begin{cases}1 & \text{for}\quad j=0\\ \dfrac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & \text{for}\quad j=1,2,\dots\end{cases}.\tag{2b}$$

is the fractional $\alpha\in\Re$ order difference of x_i.
Substituting (2) into (1) we obtain

$$Ex_{i+1}=A_\alpha x_i+\sum_{j=2}^{i+1}c_jEx_{i-j+1}+Bu_i\tag{3a}$$

where

$$A_\alpha=A+E\alpha,\ \ c_j=(-1)^j\binom{\alpha}{j}.\tag{3b}$$

It is assumed that det E = 0, but

$$\det[Ez-A_\alpha]\neq 0\ \text{ for some }\ z\in\mathrm{C},\tag{4}$$

where C is the field of complex numbers.

Assuming that for some chosen $c \in \mathrm{C}$, $\det[Ec - A_\alpha] \neq 0$ and premultiplying (3a) by $[Ec - A_\alpha]^{-1}$, we obtain

$$\overline{E}x_{i+1} = \overline{A}_\alpha x_i + \sum_{j=2}^{i+1} c_j \overline{E}x_{i-j+1} + \overline{B}u_i, \tag{5a}$$

where

$$\overline{E} = [Ec - A_\alpha]^{-1}E, \quad \overline{A}_\alpha = [Ec - A_\alpha]^{-1}A_\alpha, \quad \overline{B} = [Ec - A_\alpha]^{-1}B. \tag{5b}$$

Note that the equations (3a) and (5a) have the same solution x_i, $i \in Z_+$.

Definition 1. [4, 17] The smallest nonnegative integer q is called the index of the matrix $\overline{E} \in \Re^{n\times n}$ if

$$\mathrm{rank}\,\overline{E}^q = \mathrm{rank}\,\overline{E}^{q+1}. \tag{6}$$

Definition 2. [4, 17] A matrix $\overline{E}^D$ is called the Drazin inverse of $\overline{E} \in \Re^{n\times n}$ if it satisfies the conditions

$$\overline{E}\overline{E}^D = \overline{E}^D\overline{E}, \tag{7a}$$

$$\overline{E}^D\overline{E}\overline{E}^D = \overline{E}^D, \tag{7b}$$

$$\overline{E}^D\overline{E}^{q+1} = \overline{E}^q, \tag{7c}$$

where q is the index of $\overline{E}$ defined by (6).

The Drazin inverse $\overline{E}^D$ of a square matrix $\overline{E}$ always exists and is unique [4, 17]. If $\det\overline{E} \neq 0$ then $\overline{E}^D = \overline{E}^{-1}$. Some methods for computation of the Drazin inverse are given in [17, 19] and in the Appendix.

Theorem 1. The matrices $\overline{E}$ and $\overline{A}_\alpha$ defined by (5b) satisfy the following equalities

1. $\overline{A}_\alpha\overline{E} = \overline{E}\overline{A}_\alpha$ and $\overline{A}_\alpha^D\overline{E} = \overline{E}\overline{A}_\alpha^D$, $\overline{E}^D\overline{A}_\alpha = \overline{A}_\alpha\overline{E}^D$, $\overline{A}_\alpha^D\overline{E}^D = \overline{E}^D\overline{A}_\alpha^D$, (8a)

2. $\ker\overline{A}_\alpha \cap \ker\overline{E} = \{0\}$, (8b)

3. $\overline{E} = T\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix}T^{-1}$, $\overline{E}^D = T\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}T^{-1}$, (8c)

   $\det T \neq 0$, $J \in \Re^{n_1\times n_1}$, is nonsingular, $N \in \Re^{n_2\times n_2}$ is nilpotent, $n_1 + n_2 = n$,

4. $(I_n - \overline{E}\overline{E}^D)\overline{A}_\alpha\overline{A}_\alpha^D = I_n - \overline{E}\overline{E}^D$ and $(I_n - \overline{E}\overline{E}^D)(\overline{E}\overline{A}_\alpha^D)^q = 0.$ (8d)

Proof. Using (5b) we obtain

$$\overline{E}c - \overline{A}_\alpha = [Ec - A_\alpha]^{-1}[Ec - A_\alpha] = I_n \tag{9}$$

and

$$\overline{A}_\alpha = \overline{E}c - I_n. \tag{10}$$

Therefore

$$\overline{E}\overline{A}_\alpha = \overline{E}[\overline{E}c - I_n] = [\overline{E}c - I_n]\overline{E} = \overline{A}_\alpha\overline{E}. \tag{11}$$

The proof of the remaining equalities (8) is similar. □

Theorem 2. Let

$$P = \overline{E}\overline{E}^D, \tag{12a}$$

and

$$Q = \overline{E}^D\overline{A}_\alpha. \tag{12b}$$

Then:

5. $P^k = P$ for $k = 2,3,\ldots$ (13)
6. $PQ = QP = Q$, (14)
7. $P\overline{E}^D = \overline{E}^D$. (15)

Proof. Using (12a) we obtain

$$P^2 = \overline{E}\overline{E}^D\overline{E}\overline{E}^D = \overline{E}\overline{E}^D = P \tag{16}$$

since by (7b) $\overline{E}^D\overline{E}\overline{E}^D = \overline{E}^D$ and by induction

$$P^k = P^{k-1}P = \overline{E}\overline{E}^D\overline{E}\overline{E}^D = P^2 = P \text{ for } k = 2,3,\ldots. \tag{17}$$

Using (12) we obtain

$$PQ = \overline{E}\overline{E}^D\overline{E}^D\overline{A}_\alpha = \overline{E}^D\overline{E}\overline{E}^D\overline{A}_\alpha = \overline{E}^D\overline{A}_\alpha = Q \tag{18}$$

and

$$QP = \overline{E}^D\overline{A}_\alpha\overline{E}\overline{E}^D = \overline{E}^D\overline{E}\overline{A}_\alpha\overline{E}^D = \overline{E}^D\overline{E}\overline{E}^D\overline{A}_\alpha = \overline{E}^D\overline{A}_\alpha = Q \tag{19}$$

Using (12a), (7a) and (7b) we obtain

$$P\overline{E}^D = \overline{E}\overline{E}^D\overline{E}^D = \overline{E}^D\overline{E}\overline{E}^D = \overline{E}^D. \tag{20}$$

3. Solution to the state equation by the use of Drazin inverse

In this section the solution to the state equation (1) will be derived by the use of the Drazin inverses of the matrices $\overline{E}$ and $\overline{A}_\alpha$.

Theorem 3. The solution to the equation (5a) is given by

$$\begin{aligned} x_i &= Q^iPv + c_2Q^{i-2}Pv + c_3Q^{i-3}Pv + \ldots + 2c_{i-1}QPv + c_iPv \\ &+ \sum_{k=0}^{i-1}\overline{E}^DQ^{i-k-1}\overline{B}u_k + (P - I_n)\sum_{k=0}^{q-1}Q^k\overline{A}_\alpha^D\overline{B}u_{i+k} \end{aligned} \tag{21}$$

where Q and P are defined by (12), coefficient c_j can be computed using (3b) and $v \in \Re^n$ is arbitrary.

Proof. The system is linear thus the proof can be accomplished independently for the initial conditions and inputs. Taking into account only the first term of (21), we obtain

$$\begin{aligned} \overline{E}x_{i+1} &= \overline{E}[Q^{i+1}Pv + c_2Q^{i-1}Pv + c_3Q^{i-3}Pv + \ldots + 2c_iQPv + c_{i+1}Pv] \\ &= \overline{A}_\alpha[Q^iPv + c_2Q^{i-2}Pv + c_3Q^{i-3}Pv + \ldots + 2c_{i-1}QPv + c_iPv] \\ &= \overline{A}_\alpha x_i + \sum_{k=2}^{i}c_k\overline{E}x_{i-k+1} \end{aligned} \tag{22}$$

since by Theorems 1 and 2 conditions (8a) and (13 – 15) hold. The proof for the next two terms of (21) is similar to the proof for standard descriptor discrete-time linear systems given in [4, 17]. □

From (21) for i = 0 we have

$$x_0 = Pv + (P - I_n)\sum_{k=0}^{q-1} Q^k \bar{A}_\alpha^D \bar{B} u_k \,. \tag{23}$$

Equality (23) defines the set of consistent initial conditions X_0, $x_0 \in X_0$ for given a set of admissible inputs U_{ad}, $u_k \in U_{ad}$, $k = 0, 1,\ldots, q-1$.

If $u_k = 0$, $k = 0,1,\ldots, q-1$, then from (23) we obtain

$$x_0 = Pv \text{ and } x_0 \in \operatorname{Im} P \tag{24}$$

where $\operatorname{Im} P$ denotes the image of P.

Remark 1. The solution to the equation (5a) for $u_i = 0$, $i \in Z_+$ can be computed recurrently using the formula

$$x_i = Qx_{i-1} + \sum_{k=2}^{i} c_k P x_{i-k} \,. \tag{25}$$

Theorem 2. Let

$$\Phi_0(i) = Q^i + \sum_{k=2}^{i} c_k \bar{A}_\alpha P \,, \tag{26}$$

$$\Phi(i) = \sum_{k=0}^{i-1} \bar{E}^D Q^{i-k-1} \bar{B} \,, \tag{27}$$

where Q and P are defined by (12).

Then

$$P\Phi_0(i) = \Phi_0(i) \,, \tag{28}$$

$$P\Phi(i) = \Phi(i) \,. \tag{29}$$

Proof. Using (12) and (26), we obtain

$$P\Phi_0(i) = P\left[Q^i + \sum_{k=2}^{i} c_k \bar{A}_\alpha P\right] = Q^i + \sum_{k=2}^{i} c_k \bar{A}_\alpha P \tag{30}$$

since (14) and (13) hold.

Similarly, using (27), (14) and (15) we obtain

$$P\Phi(i) = \sum_{k=0}^{i-1} P\bar{E}^D Q^{i-k-1}\bar{B} = \sum_{k=0}^{i-1} \bar{E}^D Q^{i-k-1}\bar{B} = \Phi(i) \,. \tag{31}$$

4. Example

Consider the equation (1) with the matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 0 \\ 1 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \text{for } \alpha = 0.5 \,. \tag{32}$$

In this case we have

$$A_\alpha = A + E\alpha = \begin{bmatrix} 0.5 & 1 & 0 \\ -2 & -2.5 & 0 \\ 1 & 2 & -1 \end{bmatrix}. \tag{33}$$

The pencil of (32) is regular, since

$$\det[Ez - A_\alpha] = \begin{vmatrix} z-0.5 & -1 & 0 \\ 2 & z+2.5 & 0 \\ -1 & -2 & 1 \end{vmatrix} \tag{34}$$

We choose $c = 0$ and the matrices (5b) take the forms

$$\begin{aligned} \bar{E} &= [Ec - A_\alpha]^{-1} E = [-A_\alpha]^{-1} E \\ &= \begin{bmatrix} -0.5 & -1 & 0 \\ 2 & 2.5 & 0 \\ -1 & -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \frac{1}{0.75}\begin{bmatrix} 2.5 & 1 & 0 \\ -2 & -0.5 & 0 \\ -1.5 & 0 & 0 \end{bmatrix}, \\ \bar{A}_\alpha &= [-A_\alpha]^{-1} A_\alpha = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \end{aligned} \tag{35}$$

To compute the Drazin inverse of the matrix $\bar{E}$ we use the procedure given in the Appendix and we obtain

$$\begin{aligned} \bar{E} &= VW, \quad V = \frac{1}{0.75}\begin{bmatrix} 2.5 & 1 \\ -2 & -0.5 \\ -1.5 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\ \bar{E}^D &= V[W\bar{E}V]^{-1}W \\ &= \frac{1}{0.75}\begin{bmatrix} 2.5 & 1 \\ -2 & -0.5 \\ -1.5 & 0 \end{bmatrix} \begin{bmatrix} 7.556 & 3.556 \\ -7.111 & -3.111 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -0.5 & -1 & 0 \\ 2 & 2.5 & 0 \\ 3.5 & 4 & 0 \end{bmatrix}. \end{aligned} \tag{36}$$

and

$$\begin{aligned} P &= \bar{E}^D \bar{E} \\ &= \begin{bmatrix} -0.5 & -1 & 0 \\ 2 & 2.5 & 0 \\ 3.5 & 4 & 0 \end{bmatrix} \frac{1}{0.75}\begin{bmatrix} 2.5 & 1 & 0 \\ -2 & -0.5 & 0 \\ -1.5 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \end{aligned} \tag{37a}$$

$$\begin{aligned} Q &= \bar{E}^D \bar{A}_\alpha \\ &= \begin{bmatrix} -0.5 & -1 & 0 \\ 2 & 2.5 & 0 \\ 3.5 & 4 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0.5 & 1 & 0 \\ -2 & -2.5 & 0 \\ -3.5 & -4 & 0 \end{bmatrix}. \end{aligned} \tag{37b}$$

Using (21) for $u_k = 0,\ k = 0,1,...$ and (37) for $v = [1\ \ 2\ \ 0]^T$ we can compute

$$x_0 = Pv = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix},\quad x_1 = Qx_0 = \begin{bmatrix} 2.5 \\ -7 \\ -11.5 \end{bmatrix},$$

$$x_2 = Qx_1 + c_2 Px_0 = \begin{bmatrix} -5.688 \\ 12.625 \\ 19.563 \end{bmatrix},$$

$$x_3 = Qx_2 + c_2 Px_1 + c_3 Px_0 = \begin{bmatrix} 9.977 \\ -20.547 \\ -31.117 \end{bmatrix}, \tag{38}$$

$$x_4 = Qx_3 + c_2 Px_2 + c_3 Px_1 + c_4 Px_0 = \begin{bmatrix} -15.789 \\ 31.984 \\ 48.18 \end{bmatrix},$$

$$x_5 = Qx_4 + c_2 Px_3 + c_3 Px_2 + c_4 Px_1 + c_5 Px_0 = \begin{bmatrix} 24.58 \\ -49.324 \\ -74.068 \end{bmatrix}.$$

5. Concluding remarks

The Drazin inverse of matrices has been applied to find the solutions of the state equations of the descriptor fractional discrete-time systems with regular pencils. The equality (23) defining the set of admissible initial conditions for given inputs has been derived. Some properties of the matrices P, Q, $\Phi_0(i)$ and $\Phi(i)$ have been established (Theorem 2 and 4). The proposed method has been illustrated by a numerical example.

Comparing the presented method with the method based on the Weierstrass decomposition of the regular pencil [16], we may conclude that the proposed method is computationally attractive since the Drazin inverse of matrices can be computed by the use of well-known numerical methods [17, 19]. The presented method can be extended to the positive descriptor fractional continuous-time linear systems. An open problem is an extension of the considerations for standard and positive continuous-discrete descriptor fractional linear systems.

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A. Procedure for computation of Drazin inverse matrices

To compute the Drazin inverse $\overline{E}^D$ of the matrix $\overline{E} \in \Re^{n\times n}$ defined by (7b) the following procedure is recommended.

Procedure A.1.

Step 1. Find the pair of matrices $V \in \Re^{n\times r}$, $W \in \Re^{r\times n}$ such that

$$\overline{E} = VW, \ \text{rank}\, V = \text{rank}\, W = \text{rank}\, \overline{E} = r. \tag{A1}$$

As the r columns (rows) of the matrix V (W) the r linearly independent columns (rows) of the matrix $\overline{E}$ can be chosen.

Step 2. Compute the nonsingular matrix

$$W\overline{E}V \in \Re^{r\times r}. \tag{A2}$$

Step 3. The desired Drazin inverse matrix is given by

$$\overline{E}^D = V[W\overline{E}V]^{-1}W. \tag{A3}$$

Proof. It will be shown that the matrix (A3) satisfies the three conditions (7) of Definition 2. Taking into account that $\det WV \neq 0$ and (A1) we obtain

$$[W\overline{E}V]^{-1} = [WVWV]^{-1} = [WV]^{-1}[WV]^{-1}. \tag{A4}$$

Using (7a), (A1) and (A4) we obtain

$$\begin{aligned}\overline{E}\overline{E}^D &= VWV[W\overline{E}V]^{-1}W = VWV[WV]^{-1}[WV]^{-1}W \\ &= V[WV]^{-1}W\end{aligned} \tag{A5a}$$

and

$$\begin{aligned}\overline{E}^D\overline{E} &= V[W\overline{E}V]^{-1}WVW = V[WV]^{-1}[WV]^{-1}WVW \\ &= V[WV]^{-1}W.\end{aligned} \tag{A5b}$$

Therefore, the condition (7a) is satisfied.

To check the condition (7b) we compute

$$\begin{aligned}\overline{E}^D\overline{E}\overline{E}^D &= V[W\overline{E}V]^{-1}WVWV[W\overline{E}V]^{-1}W \\ &= V[WVWV]^{-1}WVWV[W\overline{E}V]^{-1}W \\ &= V[W\overline{E}V]^{-1}W = \overline{E}^D.\end{aligned} \tag{A6}$$

Therefore, the condition (7b) is also satisfied.

Using (7c), (A1), (A3) and (A4) we obtain

$$\begin{aligned}\overline{E}^D\overline{E}^{q+1} &= V[W\overline{E}V]^{-1}W(VW)^{q+1} \\ &= V[WV]^{-1}[WV]^{-1}WVW(VW)^q \\ &= V[WV]^{-1}W(VW)^q = VW(VW)^{q-1} \\ &= (VW)^q = \overline{E}^q\end{aligned} \tag{A7}$$

where q is the index of $\overline{E}$.

Therefore, the condition (7c) is also satisfied.