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STOKES FLOW AROUND SLOWLY ROTATING CONCENTRIC PERVIOUS SPHERES

In this paper, the problem of concentric pervious spheres carrying a fluid sink at their centre and rotating slowly with different uniform angular velocities Ω_1, Ω_2 about a diameter has been studied. The analysis reveals that only azimuthal component of velocity exists and the torque, rate of dissipated energy is found analytically in the present situation. The expression of torque on inner sphere rotating slowly with uniform angular velocity Ω_1, while outer sphere also rotates slowly with uniform angular velocity Ω_2, is evaluated. The special cases like, (i) inner sphere is fixed (i.e. $\Omega_1 = 0$), while outer sphere rotates with uniform angular velocity Ω_2, (ii) outer sphere is fixed (i.e. $\Omega_2 = 0$), while inner sphere rotates with uniform angular velocity Ω_1, (iii.) inner sphere rotates with uniform angular velocity Ω_1, while outer rotates at infinity with angular velocity Ω_2, have been deduced. The corresponding variation of torque with respect to sink parameter has been shown via figures.
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1. Introduction

Stokes flow is becoming increasingly important due to the miniaturization of fluid mechanical parts e.g., in micromechanics as well as in nanomechanics. Slow rotation of spheroids (including the disc) in an infinite fluid was first solved by **Jeffrey** [1915] using curvilinear coordinates. His approach was later extended to the spherical lens, torus, and other axisymmetric shapes. **Proudman** [1956] and **Stewartson** [1966] analyzed the dynamical properties of a fluid occupying the space between two concentric rotating spheres

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when the angular velocities of the spheres are slightly different, in other words, when the motion relative to a reference frame rotating with one of the spheres is due to an imposed azimuthal velocity which is symmetric about the equator. **Kanwal** [1960] has discussed the problem of slow steady rotation of axially symmetric bodies in a viscous fluid. **Rubinow and Keller** [1961] have considered the force on a spinning sphere which is moving through an incompressible viscous fluid by employing the method of matched asymptotic expansions to describe the asymmetric flow. **Brenner** [1961] also obtained some general results for the drag and couple on an obstacle which is moving through the fluid. **Childress** [1964] has investigated the motion of a sphere moving through a rotating fluid and calculated a correction to the drag coefficient. **Wakiya** [1967] numerically evaluated the drag and angular velocity experienced by freely rotating spheres and compared with those calculated from corresponding approximate formulae known before. **Barett** [1967] has tackled the problem of impulsively started sphere rotating with angular velocity Ω about a diameter. He modified the standard time-dependent boundary layer equation to give series solutions satisfying all the boundary conditions and gave solutions that are applicable at small times for non-zero Reynolds numbers. He found that the velocity components decay algebraically rather than exponentially at large distances. **Pearson** [1967] has presented the numerical solution for the time-dependent viscous flow between two concentric rotating spheres. He governed the motion of a pair of coupled non-linear partial differential equations in three independent variables, with singular end conditions. He also described the computational process for cases in which one (or both) of the spheres is given an impulsive change in angular velocity-starting from a state of either rest or uniform rotation. **Majumdar** [1969], has solved, by using bispherical coordinates, the non-axisymmetrical Stokes flow of an incompressible homogeneous viscous liquid in space between two eccentric spheres. It was proved that the resultant force acting upon the spheres is at right angles to the axis of rotation and the line of centres. The effect of the stationary sphere on the force and couple exerted by the liquid on the rotating sphere has been discussed and the results are compared with those of the axisymmetrical case of **Jeffrey** [1915]. **Kanwal** [1970] has considered a disk performing simple harmonic rotary oscillations about its axis of symmetry in a non-conducting viscous fluid which is at rest at infinity. **O'Neill and Majumdar** [1970] have discussed the problem of asymmetrical slow viscous fluid motions caused by the translation or rotation of two spheres. The exact solutions for any values of the ratio of radii and separation parameters are found by them.

Ranger [1971] tackled the problem of axially-symmetric flow past a rotating sphere due to a uniform stream of infinity. He has shown that leading terms for the flow consists of a linear superposition of a primary Stokes flow past a non-rotating sphere together with an anti symmetric secondary flow in the azimuthal plane induced by the spinning sphere. **Philander** [1971] presented a note on the flow properties of a fluid between concentric spheres. This note concerns the flow properties of a spherical shell of fluid when motion is forced across the equator. The fluid under consideration is contained between two concentric spheres which rotate about a diameter with angular velocity Ω. The consequences of the forcing motion across the equator are explored in his work. **Cooley** [1971] has investigated the problem of fluid motion generated by a sphere rotating close to a fixed sphere about a diameter perpendicular to the line of centres in the case when the motion is sufficiently slow to permit the linearization of the Navier-Stokes equations by neglecting the inertia terms. He used a method of matched asymptotic expansions to find asymptotic expressions for the forces and couples acting on the spheres as the minimum clearance between them tends to zero. In his paper, the forces and couples are shown to have the form $a_0 \ln \varepsilon + a_1 + o(\varepsilon \ln \varepsilon)$, where ε is the ratio of the minimum clearance between the spheres and the radius of the rotating sphere and where a_0 and a_1 are found explicitly. **Munson and Joseph** [1971, part 1 and part 2] have obtained the high order analytic perturbation solution for the viscous incompressible flow between concentric rotating spheres. In second part of their analysis, they have applied the energy theory of hydrodynamic stability to the viscous incompressible flow of a fluid contained between two concentric spheres which rotate about a common axis with prescribed angular velocities. **Riley** [1972] has discussed the thermal effects on slow viscous flow between rotating concentric spheres. **Takagi** [1974a] has considered the flow around a spinning sphere moving in a viscous fluid. He solved the Navier-Stokes equations, using the method of matched asymptotic expansions for small values of the Reynolds number. With the solution, the force and torque on the sphere are computed, and he found that the sphere experiences a force orthogonal to its direction of motion and that the drag is increased in proportion to the square of the spin velocity. **Takagi** [1974b] has studied the Stokes flow for the case in which two solid spheres in contact are steadily rotating with different angular velocities about their line of centres. For the case of two equal spheres, one of which is kept rotating with angular velocity ω while the other is left free, he found that the latter will rotate with angular velocity $\omega/7$. **Munson and Menguturk** [1975, part 3] have studied the stability of flow of a viscous incompressible fluid between a stationary outer sphere and rotating inner sphere theoretically and experimentally. **Wimmer** [1976] has provided some experimental results on incompressible viscous fluid flow in the gap between two concentric rotating spheres. **Takagi** [1977] further studied the problem of steady flow which is induced by the slow rotation of a solid sphere immersed in an infinite incompressible viscous fluid, on the basis of Navier-Stokes equations. He obtained the solution in the form of power series with respect to Reynolds number. **Drew** [1978] has found the force on a small sphere translating relative to a slow viscous flow to order of the $^1/_2$ power of Re for two different fluid flows far from the sphere, namely pure rotation and pure shear. For pure rotation, the correction of this order to the Stokes drag consists of an increase in the drag. **Kim** [1980] has calculated the torque and frictional force exerted by a viscous fluid on a sphere rotating on the axis of a circular cone of arbitrary vertex angle about an axis perpendicular to the cone axis in the Stokes approximation. **Dennis et al.** [1981] have investigated the problem of viscous incompressible, rotationally symmetric flow due to the rotation of a sphere with a constant angular velocity about a diameter. The solutions of the finite-difference equations are presented for Reynolds number ranging from 1.0 to 5000. **Davis** and **Brenner** [1986] have used the matched asymptotic expansion methods to solve the problem of steady rotation of a tethered sphere at small, non-zero Reynolds numbers. They obtained first order Taylor number correction to both the Stokes-law drag and Kirchhoff's law couple on the sphere for Rossby numbers of order unity. **Gagliardi** [1987] has developed the boundary conditions for the equations of motion for a viscous incompressible fluid in a rotating spherical annulus. The solution of the stream and circumferential functions were obtained in the form of a series of powers of the Reynolds number. Transient profiles were obtained for the dimensional torque, dimensionless angular velocity of the rotating sphere, and the dimensionless angular momentum of the fluid. **Marcus and Tuckerman** [1987, part 1 and 2] have computed numerically the steady and translation simulation of flow between concentric rotating spheres. **O'Neill and Yano** [1988] derived the boundary condition at the surfactant and substrate fluids caused by the slow rotation of a solid sphere which is partially submerged in the substrate fluid. **Yang** et al [1989] have provided the numerical schemes for the problem of the axially symmetric motion of an incompressible viscous fluid in an annulus between two concentric rotating spheres. **Gagliardi** et al. [1990] reported the study of the steady state and transient motion of a system consisting of an incompressible, Newtonian fluid in an annulus between two concentric, rotating, rigid spheres. They solved the governing equations for the variable coefficients by separation of variables and Laplace Transform methods. They presented the results for the stream function, circumferential function, angular velocity of the spheres and torque coefficient as a function of time for various values of the dimensionless system parameters. **Ranger** [1996] has found an exact solution of the Navier-Stokes equations for the axi-symmetric motion (with swirl) representing exponentially time-dependent decay of a solid sphere translating and rotating in a viscous fluid relative to a uniform stream whose speed also decays exponentially with time. He also described a similar solution for the two-dimensional analogue where the sphere is replaced by a circular cylinder of infinite length. **Tekasakul** et al. [1998] have studied the problem of the rotatory oscillation of an axi-symmetric body in an axi-symmetric viscous flow at low Reynolds numbers. They evaluated numerically the local stresses and torques on a selection of free, oscillating, axi-symmetric bodies in the continuum regime in an axi-symmetric viscous incompressible flow. **Datta and Srivastava** [2000] have tackled the problem of slow rotation of a sphere with fluid source at its centre in a viscous fluid. In their investigation, it was found that the effect of fluid source at the centre is to reduce the couple on slowly rotating sphere about its diameter. **Kim and Choi** [2002] conducted the numerical simulations for laminar flow past a sphere rotating in the streamwise direction, in order to investigate the effect of the rotation on the characteristics of flow over the sphere.

Tekasakul and Loyalka [2003] have investigated the rotary oscillations of several axi-symmetric bodies in axi-symmetric viscous flows with slip. A numerical method based on the Green's function technique is used and analytic solutions for local stress and torque on spheres and spheroids as function of the frequency parameter and the slip coefficients are obtained. They have analysed that in all cases, slip reduces stress and torque, and increasingly so with the increasing frequency parameter. **Liu** et al. [2004] have developed a very efficient numerical method based on the finite difference technique for solving time-dependent non-linear flow problems. They have applied this method to study the unsteady axisymmetric isotherm flow of an incompressible viscous fluid in a spherical shell with a stationary inner sphere and a rotating outer sphere. **Ifidon** [2004] numerically investigated the problem of determining the induced steady axially symmetric motion of an incompressible viscous fluid confined between two concentric spheres, with the outer sphere rotating with constant angular velocity and the inner sphere fixed for large Reynolds numbers. **Davis** [2006] obtained the expression for force and torque on a rotating sphere close to and within a fluid-filled rotating sphere. **Marcello** [2008] has introduced new exact analytic solutions for the rotational motion of an axially symmetric rigid body having two equal principal moments of inertia and subjected to an external torque which is constant in magnitude. Recently, **Srivastava et al.** [2011] have studied the effect of viscous fluid around slowly rotating sphere with sink at its centre in which they concluded that the effect of sink at the centre is to reduce the couple.

In the present paper, the problem of slow rotation of concentric spheres, both assumed to be pervious, with a sink at their centre has been tackled. If the strength '-Q' of the sink were of the same order as the angular velocity Ω of rotating spheres, the inertia terms could still be neglected, and the total flow then consists of only the source solution superimposed on the Stokes solution. Therefore, in this case the Stokes drag and torque are not affected by the sink. Also, if Q is large enough so that $Q\Omega$ is not negligible, the inertia terms, being non-linear, cannot be altogether omitted. The Navier-Stokes equation, can still be linearized by assuming that the velocity perturbation in the source flow on account of the Stokes flow is small, so that the terms containing square of angular velocity (i.e. of order Ω^2) can be neglected. This assumption is justifiable at least in the vicinity of the spheres where the Stokes approximation is valid. The present problem corresponds to the problem of Stokes flow past a sphere with source at its centre investigated by **Datta** [1974] and slow rotation of sphere with source at its centre in a viscous fluid investigated by **Datta & Srivastava** [2000], **Srivastava et al.** [2011], the results of which have found engineering application mainly in investigation of the diffusiophoresis target efficiency for an evaporating or condensing drop [**Placek and Peters**, 1980].

2. Formulation of the problem

Let us consider two pervious spheres of radius 'a' and 'b' (where b >a) with sink of strength '-Q' at its centre generating radial inward flow around it in an infinite expense of incompressible fluid of density ρ and kinematic viscosity ν. The spheres are also made to rotate with small steady angular velocities Ω_1 and Ω_2 so that terms of an $o(\Omega^2)$ may be neglected but terms of $o(Q\Omega)$ retained. The motivation of this formulation has been taken from the author's previous works [**Datta and Srivastava**, 2000, **Srivastava et al.**, 2011] due to the fact that body geometry has not been changed, although the two concentric spheres are rotating slowly with different angular velocities instead of only one. The governed equations of motion will remain the same and provide the new solutions under the defined boundary conditions. The main aim of the present formulation is to study the effect of sink at the centre of slowly rotating spheres over torque to maintain the motion.

The motion is governed by steady Navier-Stokes equations

$$\mathbf{u}\ .grad\,\mathbf{u} = -\left(\frac{1}{\rho}\right) grad\, p\ + \nu\,\nabla^2\mathbf{u} \tag{1}$$

and continuity equation

$$div\mathbf{u}\ = 0, \tag{2}$$

together with no-slip boundary condition

$$\mathbf{u}\ = a\Omega\hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_r, \quad \text{on the inner sphere } r = a, \tag{3a}$$

$$\mathbf{u} = b\Omega\hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_r, \quad \text{on the outer sphere } r = b, \tag{3b}$$

and the condition of vanishing of velocity at far-off points

$$\mathbf{u} = 0 \quad \text{as} \quad r \to \infty. \tag{4}$$

The velocity considered in (2.3) is not posed for complete flow, but only for the difference between full velocity and the velocity induced by the sink at centre. In the above equations, symbols $\mathbf{u}$, p , ρ , ν stands for velocity , pressure, density and kinematic viscosity and unit vectors $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_r$ are along x-axis and radial direction. It will be convenient to work in spherical polar coordinates (r, θ, ϕ) with x-axis as the polar axis. We non-dimensionalize the space variables by a , velocity by a Ω , and pressure by $\rho\ \nu\ \Omega$. Moreover, the symmetry of the problem and the boundary conditions ensure that velocity components $v_r = v_\theta = 0$, and then we may express the non-dimensional velocity vector $\mathbf{u}$ as

$$\mathbf{u} = -\frac{Q}{a^2 r^2}\hat{e}_r + v_\varphi(r\,,\,\theta)\,\hat{e}_\varphi \tag{5}$$

and pressure as

$$p = \rho\nu\Omega[p_0(r) + p_1(r,\theta)]. \tag{6}$$

By introducing the expressions (5) and (6) in equation (1), the azimuthal component v_ϕ is seen to satisfy the equation

$$\nabla^2 v_\varphi - \frac{v_\varphi}{r^2 \sin\theta} = -\frac{s}{r^3}\frac{\partial}{\partial r}\left(r\, v_\varphi\right), \tag{7}$$

where $s = \dfrac{Q}{\nu\, a}$ is the sink parameter.

The above equation is to be solved under the boundary conditions

$$\left.\begin{aligned} &v_\varphi = \sin\theta \quad \text{at} \quad r = 1 \text{ (non − dimensional equation of spheres)} \\ &\text{and} \\ &v_\varphi \to 0 \quad \text{as} \quad r \to \infty. \end{aligned}\right\} \tag{8}$$

3. Solution

We take the trial solution as

$$v_\phi = r\omega(r)\sin\theta, \tag{9}$$

substituting this value of v_ϕ into equation (7), we get, after some calculation and adjustment

$$\frac{d}{dr}\left[r^4\frac{d\omega}{dr} + s\,r^2\omega\right] = 0, \tag{10}$$

and the boundary conditions (8) in non-dimensional form become

$$\omega = 1 \quad \text{at} \quad \mathrm{r} = 1 (\text{i.e. on the surface}) \tag{11}$$

and

$$\omega \to 0 \quad \text{as} \quad \mathrm{r} \to \infty. \tag{12}$$

The above boundary conditions may also be express in dimensional form as

$$\omega = \Omega_1 \quad \text{at} \quad \mathrm{r} = \mathrm{a} (\text{i.e. on the inner sphere}) \tag{13}$$

and

$$\omega \to 0 asr \to \infty. \tag{14}$$

On integration of equation (10), we get the solution in non-dimensional form as

$$\omega(r) = \frac{A}{s^3}\left[\frac{s^3}{r^2} + 2\frac{s}{r} + 2\right] + B\, e^{\frac{s}{r}} \tag{15}$$

and in dimensional form as

$$\omega(r) = \frac{A}{s^3}\left[\frac{s^2a^2}{r^2} + 2\frac{sa}{r} + 2\right] + B\, e^{\frac{sa}{r}}, \tag{16}$$

where A and B are constants of integration which can be obtained by applying boundary conditions (13) and (14) as

$$\frac{A}{s^3} = \Omega_1\left[s^2 + 2s + 2 - 2e^s\right]^{-1}$$

and

$$B = -2\Omega_1\left[s^2 + 2s + 2 - 2e^s\right]^{-1}.$$

Substituting the values of constants A and B in (16), we get the expression of angular velocity ω(r) in dimensional form as

$$\omega(r) = \frac{A}{s^3}\left[\frac{s^2a^2}{r^2} + 2\frac{sa}{r} + 2\right] + B\, e^{\frac{sa}{r}}$$

or

$$\omega(r) = \Omega_1\left[\frac{s^2a^2}{r^2} + 2\frac{sa}{r} + 2 - e^{\frac{sa}{r}}\right]\left[s^2 + 2s + 2 - 2e^s\right]^{-1} \tag{17}$$

and consequently, with the help of (9), the expression for azimuthal component of velocity v_ϕ comes out to be in dimensional form as

$$v_\varphi = r\omega(r)\sin\theta = \Omega_1 r\,\sin\theta\left[\frac{s^2a^2}{r^2} + 2\frac{sa}{r} + 2 - e^{\frac{sa}{r}}\right]\left[s^2 + 2s + 2 - 2e^s\right]^{-1}. \tag{18}$$

4. Torque on inner sphere rotating with uniform angular velocity Ω_1 (when outer sphere is also rotating with different uniform angular velocity Ω_2)

If there exists an external concentric pervious sphere of radius b (b>a), rotating with small angular velocity Ω_2 i.e., the boundary conditions for this situation will be

$$\omega = \Omega_2 \quad \text{at} \quad \text{r} = \text{b (at outer surface)} \tag{19}$$

and

$$\omega = \Omega_1 \quad \text{at} \quad \text{r} = \text{a (at inner surface).} \tag{20}$$

by using the above boundary conditions, in equation (16), the constant of integration A and B comes out to be

$$-\frac{A}{s^3} = \frac{\Omega_1 - \Omega_2 e^{s\left(1-\frac{a}{b}\right)}}{\left[\left(\frac{s^2a^2}{b^2} + 2\frac{sa}{b} + 2\right) e^{s\left(1-\frac{a}{b}\right)} + \left(-s^2 - 2s - 2\right)\right]} \tag{21}$$

and

$$B = e^{-\frac{sa}{b}} \left[\frac{\Omega_1 \left\{\frac{s^2a^2}{b^2} + 2\frac{sa}{b} + 2\right\} + \Omega_2 \left\{-s^2 - 2s - 2\right\}}{e^{s\left(1-\frac{a}{b}\right)} \left\{\frac{s^2a^2}{b^2} + 2\frac{sa}{b} + 2\right\} + \left\{-s^2 - 2s - 2\right\}} \right]. \tag{22}$$

The expression of angular velocity ω(r) can be written with the help of equation (16)

$$\omega(r) = \frac{A}{s^3}\left[\frac{s^2a^2}{r^2} + 2\frac{sa}{r} + 2\right] + B\, e^{\frac{sa}{r}},$$

where A and B are given in (21) and (22). On differentiating the function ω(r), we have

$$\frac{d\omega}{dr} = \frac{A}{s^3}\left[-\frac{2s^2a^2}{r^3} - \frac{2sa}{r^2}\right] + B\, e^{\frac{sa}{r}}\left(-\frac{sa}{r^2}\right) \quad ,$$

the value of $\dfrac{d\omega}{dr}$ at r = a can be written as

$$\left(\frac{d\omega}{dr}\right)_{r=a} = \frac{1}{a}\left[-\frac{2A}{s^3}\left(s^2 - s\right) - Bs\, e^{s}\right]. \tag{23}$$

The moment of force p_ϕ is p_ϕ. r sin θ, where $p_\phi = \mu$. r sin θ. $\dfrac{d\omega}{dr}$, is the only non-vanishing component of force p . If N is the torque on the sphere of radius a, then by using (23), we have

$$N = \int_0^\pi \left(p_\varphi . r \sin\theta\right)_{r=a} dS$$

$$= \int_0^{\pi} \left(\mu r \, \sin\theta \, \frac{d\omega}{dr} r \, \sin\theta \right)_{r=a} . \, (2\pi \, a \, \sin\theta \, . \, a \, d\theta)$$

$$= \frac{8}{3}\pi a^3 \mu \left[-\frac{2A}{s^3}\left(s^2 + s\right) + Be^s \right]$$

$$= \frac{8}{3}\pi a^3 \mu \left[\begin{array}{l} 2\left(s^2+s\right)\left\{\Omega_1 - \Omega_2 s e^{s\left(1-\frac{a}{b}\right)}\right\} \\ +se^{s\left(1-\frac{a}{b}\right)}\left\{-\Omega_1\left(\frac{s^2a^2}{b^2} + 2\frac{sa}{b} + 2\right) + \Omega_2\left(s^2+2s+2\right)\right\} \end{array} \right] \times \left[\left\{ -e^{s\left(1-\frac{a}{b}\right)} \left(\frac{s^2a^2}{b^2} + 2\frac{sa}{b} + 2 \right) + \left(s^2+2s+2\right) \right\} \right]^{-1} . \tag{24}$$

If inner and outer spheres are rotating with same angular velocities, i.e. $\Omega_1 = \Omega_2 = \Omega$ and b = 2a, then from (24), torque coefficient (normalizing with $8\pi\mu a^3\Omega$, torque on sphere having radius ‘a’) is given by

$$\frac{N}{8\pi\mu a^3\Omega} = \frac{1}{3}\left[8\left(s+s^2\right) - s^2 e^{\frac{s}{2}}\left(4+5s\right)\right]\left[4\left(2+2s+s^2\right) - \left(s^2+4s+8\right)e^{\frac{s}{2}}\right]^{-1} \tag{24a}$$

If inner and outer spheres are rotating with different angular velocities, i.e. $\Omega_2 = 2\Omega_1$ and b = 2a, then from (24), torque coefficient (normalizing with $8\pi\mu a^3\Omega_1$, torque on sphere having radius ‘a’) is given by

$$\frac{N}{8\pi\mu a^3\Omega_1} = \frac{1}{3}\left[8s\left(1+s+e^{\frac{s}{2}}\right) - s^2 e^{\frac{s}{2}}\left(4+9s\right)\right]\left[4\left(2+2s+s^2\right) - \left(8+4s+s^2\right)e^{\frac{s}{2}}\right]^{-1} \tag{24b}$$

The rate of dissipated energy is given by $N\Omega_1$, where the value of N is given in equation (24), (24a, b).

5. Torque on outer sphere rotating with uniform angular velocity Ω_2 (when inner sphere is fixed, i.e. $\Omega_1 = 0$)

The expression for angular velocity ω(r) is given by (16)

$$\omega(r) = \frac{A}{s^3}\left[\frac{s^2a^2}{r^2} + 2\frac{sa}{r} + 2\right] + B\,e^{\frac{sa}{r}}.$$

Now we use the following boundary conditions

$$\omega(r) = \Omega_2 \quad \text{on surface} \quad r = b \tag{25}$$

and

$$\omega(r) \to 0 \quad \text{as} \quad r \to \infty. \tag{26}$$

Under these boundary conditions, the values of constant A and B can be obtained as follows

$$\frac{A}{s^3} = \Omega_2 \left[\frac{s^2a^2}{b^2} + \frac{2sa}{b} - 2e^{\frac{sa}{b}} + 2 \right]^{-1} \tag{27}$$

and

$$B = -2\Omega_2 \left[\frac{s^2a^2}{b^2} + \frac{2sa}{b} - 2e^{\frac{sa}{b}} + 2 \right]^{-1}. \tag{28}$$

Now, the expression for derivative of angular velocity at r = b comes out to be

$$\left[\frac{d}{dr}\omega(r)\right]_{r=b} = \left[\frac{A}{s^3}\left(-\frac{2s^2a^2}{r^3} - \frac{2sa}{r^2}\right) + B\left(-\frac{sa}{r^2}\right)e^{\frac{sa}{r}}\right]_{r=b}$$

$$= -\frac{1}{b}\left[\frac{2A}{s^3}\left(\frac{s^2a^2}{b^2} + \frac{sa}{b}\right) + B\frac{sa}{b}e^{\frac{sa}{b}}\right],$$

which reduces in final form by (27) and (28)

$$= -\frac{2\Omega_2}{b}\left[\frac{s^2a^2}{b^2} + \frac{sa}{b} - \frac{as}{b}e^{\frac{sa}{b}}\right]\left[\frac{s^2a^2}{b^2} - \frac{2sa}{b} - 2e^{\frac{sa}{b}} + 2\right]^{-1}. \tag{29}$$

Hence, torque N on the outer sphere in the presence of inner sphere

$$N = \int_0^{\pi} \left(\mu\,.\,r\,\sin\theta\,.\,\frac{d\omega}{dr}\,.\,r\,\sin\theta\right)_{r=b} (2\pi b\,\sin\theta\,.\,b\,d\theta)$$

$$= \frac{8}{3}\pi\, b^3 \mu \left(\frac{d}{dr}\omega(r)\right)_{r=b},$$

by using (29), it reduces to

$$= \frac{16}{3}\pi\, b^3 \mu\, \Omega_2 \left[\frac{s^2a^2}{b^2} + \frac{sa}{b} - \frac{sa}{b}e^{\frac{sa}{b}}\right]\left[\frac{s^2a^2}{b^2} + 2\frac{sa}{b} - 2e^{\frac{sa}{b}} + 2\right]^{-1}. \tag{30}$$

If b = 2a, then from (30), torque coefficient (normalizing with $8\pi\mu a^3\Omega_2$, torque on sphere having radius ‘a’) is given by

$$\frac{N}{8\pi\mu a^3\Omega_2} = \frac{2}{3}\left[2s + s^2 - 2se^{\frac{s}{2}}\right]\left[s^2 + 4s + 8 - 8e^{\frac{s}{2}}\right]^{-1} \tag{31}$$

The expression for rate of dissipated energy will be $N\Omega_2$, where N is given in equation (30) and (31).

6. Particular Cases

Case 1. We consider the outer spherical surface to be fixed i.e., $\Omega_2 = 0$, then in this case, by (24), we have, torque on the inner sphere rotating with angular velocity Ω_1 as

$$\begin{aligned} N = {} & \frac{8}{3}\pi a^3\mu\Omega_1\left[2\left(s^2+s\right)-s\left(\frac{s^2a^2}{b^2}+2\frac{sa}{b}+2\right)e^{s\left(1-\frac{a}{b}\right)}\right] \\ & \left[\left(-s^2-2s-2\right)+\left(\frac{s^2a^2}{b^2}+2\frac{sa}{b}+2\right)e^{s\left(1-\frac{a}{b}\right)}\right]^{-1}. \end{aligned} \tag{32a}$$

If b = 2a, then from (32a), torque coefficient (normalizing with $8\pi\mu a^3\Omega_1$, torque on sphere having radius 'a') is given by

$$\begin{aligned} \frac{N}{8\pi\mu a^3\Omega_1} = {} & \frac{1}{3}\left[8\left(s+s^2\right)-s\left(s^2+4s+8\right)e^{\frac{s}{2}}\right] \\ & \left[-4\left(2+2s+s^2\right)+\left(8+4s+s^2\right)e^{\frac{s}{2}}\right]^{-1} \end{aligned} \tag{32b}$$

Now, on shifting the solid outer spherical body having radius b (b>a) to infinity i.e., b→ ∞ , then $e^{s\left(1-\frac{a}{b}\right)} \to e^s$ and by (32a), we can have the expression for torque on slowly rotating inner sphere of radius 'a' alone and given by (32b) as

$$N = \frac{16}{3}\pi\mu a^3\Omega_1\frac{\left[s^2+s-se^s\right]}{\left[s^2+2s+2-2e^s\right]}, \tag{33}$$

which match with the expression of torque obtained by **Srivastava et al.** [2011] for slowly rotating pervious sphere of radius 'a' rotating with slow uniform angular velocity Ω_1 with sink at the centre and further reduces to classical one $8\pi\mu a^3\Omega_1$ for s = 0 (i.e. in the absence of sink at the center).

Case 2. If the inner sphere rotates with uniform angular velocity Ω_1, while outer rotates with uniform angular velocity Ω_2 at infinity, i.e. b → ∞, then by expression (24), we have the torque on inner sphere as

$$N = \frac{8}{3}\pi\mu a^3\frac{\left[2\Omega_1\left(s^2+s-se^s\right)+\Omega_2 se^s\left(2-s^2\right)\right]}{\left[s^2+2s+2-2e^s\right]}. \tag{34a}$$

If $\Omega_2 = 2\Omega_1$, then from (34a), torque coefficient (normalizing with $8\pi\mu a^3\Omega_1$, torque on sphere having radius 'a') is given by

$$\frac{N}{8\pi\mu a^3\Omega_1} = \frac{2}{3}\left[s+s^2+se^s\left(1-s^2\right)\right]\left[2+2s+s^2-2e^s\right]^{-1} \tag{34b}$$

If inner and outer spheres are rotating with same angular velocities, i.e. $\Omega_1 = \Omega_2 = \Omega$ and then from (34a), torque coefficient (normalizing with $8\pi\mu a^3\Omega$, torque on sphere having radius 'a') is given by

$$\frac{N}{8\pi\mu a^3\Omega} = \frac{1}{3}\left[2s + 2s^2 - s^3e^s\right]\left[2 + 2s + s^2 - 2e^s\right]^{-1} \tag{34c}$$

Case 3. If we consider the limiting situation as a → b and $\Omega_1 \to \Omega_2$, then we have the expression for torque on slowly rotating sphere having radius 'a' by (30)

$$N = \frac{16}{3}\pi\, a^3\, \mu\, \Omega_2\left[s^2 + s - se^s\right]\left[s^2 + 2s + 2 - 2e^s\right]^{-1}, \tag{35}$$

which agrees with that given in the paper of **Srivastava et al.** [2011] and further reduces to the classical one $M_0 = 8\pi\mu a^3\Omega_1$ for s = 0 (i.e. in the absence of sink at the center).

7. Numerical Discussion

Variation of angular velocity ω(r) [equation (17)] with respect to 'r' for various values of sink parameter 's' are shown in Figure 1. For increasing values of 'r' , the value of angular velocity ω(r) gets dampened steadily and reduces ultimately to zero for specific values of sink parameter 's'. It is interesting to note here that for values of 'r' ($0 \le r \le 1$), ω(r) sharply comes down to 1, and then slowly dies down to zero for values of r >1. Torque coefficient [equation (24a) and (24b)] for inner rotating sphere decreases with respect to increasing values of sink parameter 's' in the presence of outer sphere having radius 2a and rotating with same angular velocity in first case and $2\Omega_1$ in the second case. Both these variations are depicted in Figures 2 (i) and 2 (ii). Torque coefficient [equation (31)] on outer rotating sphere (in the presence of fixed inner wall) having radius 2a increases from 1 (for s=0) to ∞ with respect to increasing values of sink parameter 's'. Torque coefficient [equation (32b)] on inner wall (in the presence of fixed outer wall having radius 2a) increases from 1.2 (for s=0) to ∞ with respect to increasing values of sink parameter 's'. Further, torque coefficient [equation (34b)] for inner sphere (when outer wall is present at far distance and rotating with angular velocity $2\Omega_1$) increases with respect to increasing values of sink parameter 's'. These variations are depicted in Figures 3, 4 and 5. In all calculations of torque coefficient in different situations, normalization is done via classical value of torque, $8\pi\mu a^3\Omega$, on rotating sphere having radius 'a'.

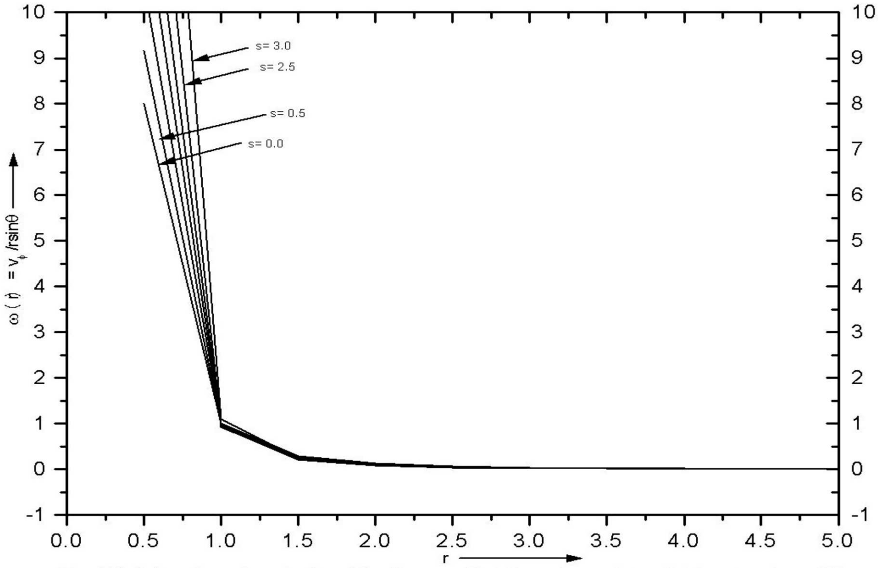

Fig. 1. Variation of angular velocity ω(r) with respect to 'r' for various values of sink parameter s=Q/νa

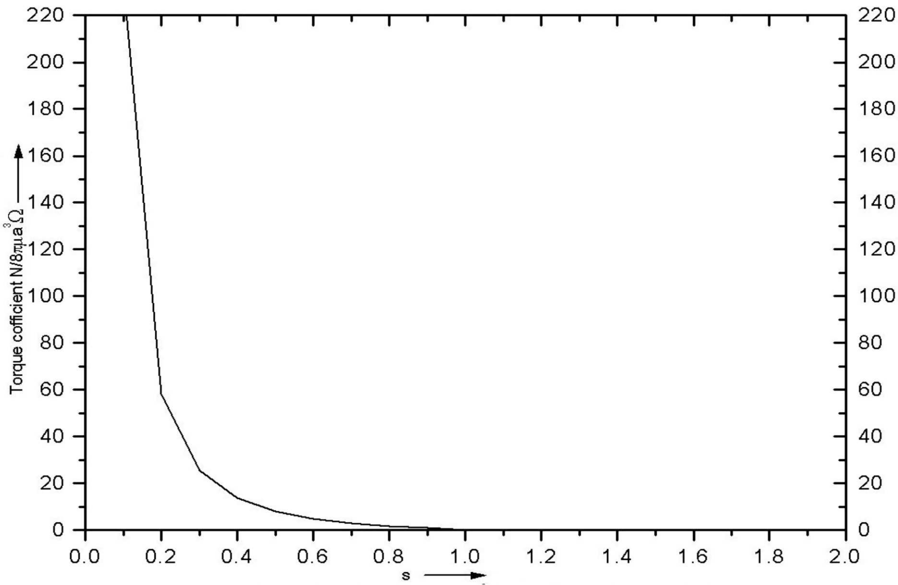

Fig. 2. (i) Variation of Torque cofficient N/8$\pi\mu a^3\Omega$ with respect to sink parameter 's'

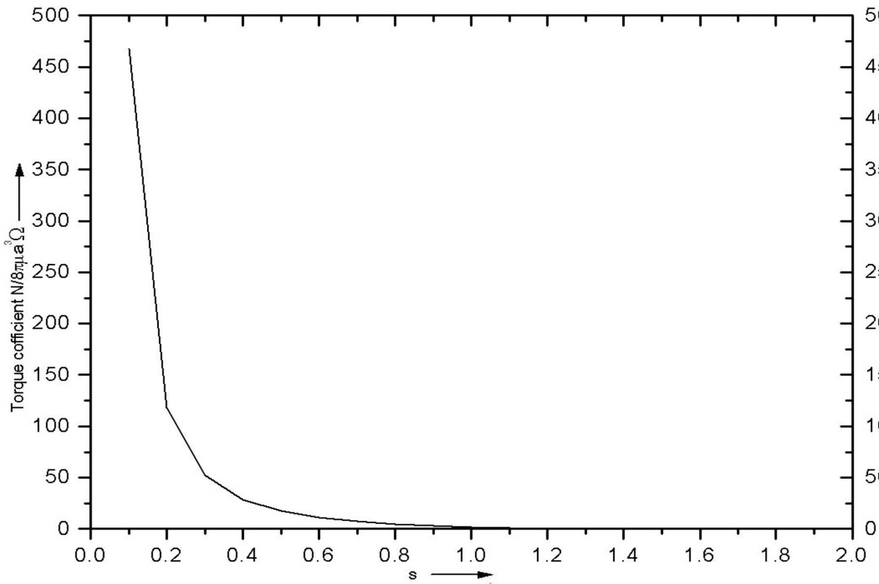

Fig. 2 (ii) Variation of Torque cofficient $N/8\pi\mu a^3\Omega$ with respect to sink parameter 's'

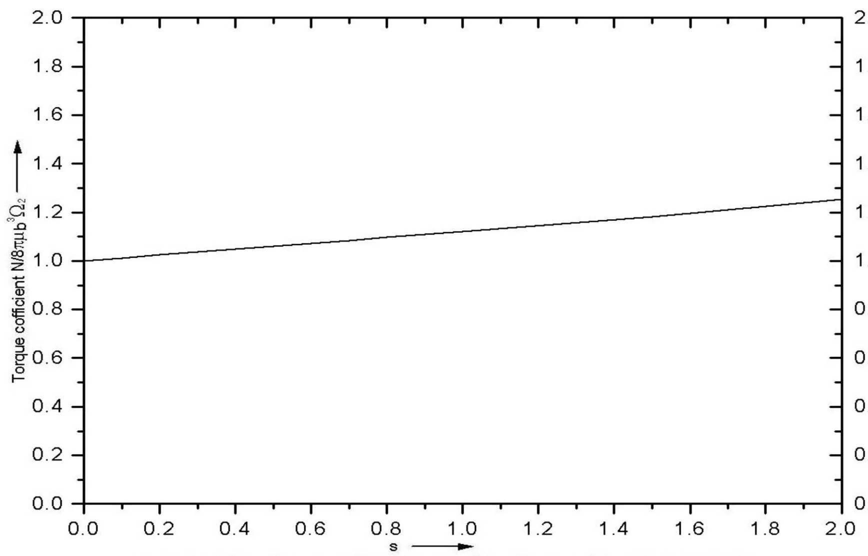

Fig. 3. Variation of Torque cofficient $N/8\pi\mu b^3\Omega_2$ with respect to sink parameter 's'

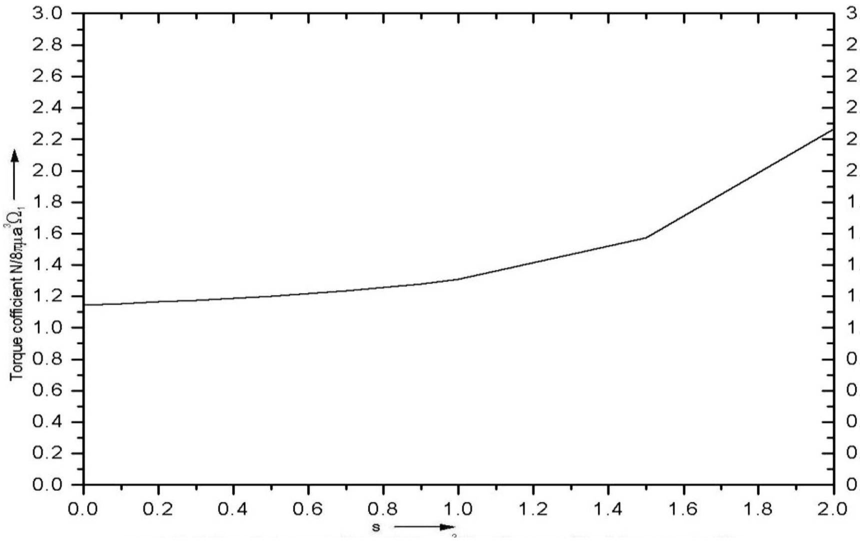

Fig. 4. Variation of Torque cofficient $N/8\pi\mu a^3\Omega_1$ with respect to sink parameter 's'

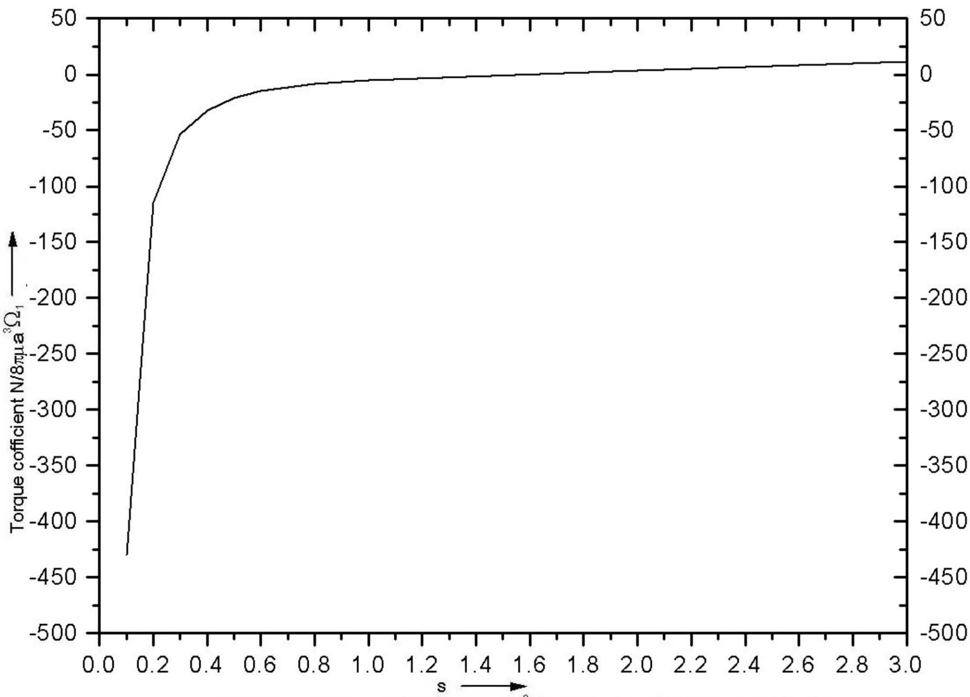

Fig. 5. Variation of Torque cofficient $N/8\pi\mu a^3\Omega_1$ with respect to sink parameter 's'

8. Conclusion

The problem of slow rotation of two concentric pervious spheres having fluid sink at the centre in viscous fluid is solved. The expressions for torque in general case (3.16, 3.22) and in cases (4.1 to 4.3) are calculated and expected to be new and never seen in the literature. It has been observed that rotation of either wall create interference for the rotation of other wall resulting in a decrease of torque with respect to increasing sink parameter. While, on the other hand, torque on either wall increases with respect to increasing values of sink parameter when the other wall is kept fixed. The results found here may be very useful in the study of evaporating or condensing spherical drop in nature.

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Przepływ stokesowski wokół powolnie wirujących koncentrycznych kul przepuszczalnych

S t r e s z c z e n i e

W artykule rozważa się problem koncentrycznych kul przepuszczalnych, ze zlewem płynu w centrum, które wirują powoli wokół średnicy z jednostajnymi prędkościami kątowymi Ω_1 i Ω_2. Analiza wykazała, że istnieje tylko azymutalny składnik prędkości, a moment obrotowy i szybkość rozpraszania energii są w istniejących warunkach wyznaczane analitycznie. Wyprowadzono wyrażenie na moment obrotowy na powierzchni wewnętrznej kuli powolnie wirującej z jednostajną prędkością kątową Ω_1, podczas gdy kula zewnętrzna także powolnie wiruje z jednostajną prędkością kątową Ω_2. Zbadano także przypadki szczególne, takie jak: (i) kula wewnętrzna jest nieruchoma (tzn. $\Omega_1 = 0$), podczas gdy kula zewnętrzna wiruje z jednostajną prędkością kątową Ω_2, (ii) kula zewnętrzna jest nieruchoma (tzn. $\Omega_2 = 0$), podczas gdy kula wewnętrzna wiruje z jednostajną prędkością kątową Ω_1, (iii) kula wewnętrzna wiruje z jednostajną prędkością kątową Ω_1, podczas gdy kula zewnętrzna wiruje w nieskończonej odległości z prędkością kątową Ω_2. Na wykresach przedstawiono zależności między zmianami momentu obrotowego a parametrami zlewu.