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Optimal choice of integral parameter in a process of solving the inverse problem for heat equation

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Abstract Inverse boundary problem for cylindrical geometry and unsteady heat conduction equation was solved in this paper. This solution was presented in a convolution form. Integration of the convolution was made assuming the distribution of temperature T on the integration interval $\langle t_i, t_i + \Delta t\rangle$ in the form $T(x,t) = T(x,t_i)\,\Theta + T(x,t_i+\Delta t)\,(1-\Theta)$, where $\Theta \in (0,1)$. The influence of value of the parameter Θ on the sensitivity of the solution to the inverse problem was analysed. The sensitivity of the solution was examined using the SVD decomposition of the matrix A of the inverse problem and by analysing its singular values. An influence of the thermocouple installation error and stochastic error of temperature measurement as well as the parameter Θ on the error of temperature distribution on the edge of the cylinder was examined.

Keywords: Inverse problem; Sensitivity of the solution; Heat conduction

Nomenclature

c	–	specific heat, J/kg K
C	–	integration constant
cond$A$	–	condition number of the matrix A
dp	–	values calculated using the direct problem
e	–	Eulerian number
g	–	distance of the thermocouple from the edge of the cylinder, m
I	–	identity (square) matrix of size N

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I_0, I_1, I_{-n}	–	modified Bessel functions of the first kind
ip	–	values calculated using the inverse problem
J_0, J_1, J_n	–	Bessel functions of the first kind
L, L^{-1}	–	Laplace transform, inverse Laplace transform
r	–	radius, m
random	–	values calculated with stochastic disturbances of the temperature measurement
rank$A$	–	rank of the matrix A
s	–	complex variable
t	–	time, s
T	–	temperature, °C
U, V^*	–	unitary matrix in the SVD decomposition
$+$	–	values calculated with the thermocouple installation error by δr^*into the edge of the cylinder
$*$	–	convolution of function
$< >$	–	closed interval

Greek symbols

β	–	coefficient in assumed function of temperature on the edge of the cylinder
δ	–	absolute error
Δ	–	difference
ϑ	–	temperature in nondimensional coordinates
Θ	–	coefficient used during integration $\Theta \in (0, 1)$
λ	–	heat conduction coefficient of the cylinder, W/m K
ξ	–	radius in nondimensional coordinates
ρ	–	density, kg/m^3
σ	–	singular values of the matrix A
Σ	–	diagonal matrix in the SVD decomposition
τ	–	time in nondimensional coordinates

Subscripts

0	–	initial time, for $t = 0$
max	–	maximum while heating
w	–	at assumed constant temperature on the edge of the cylinder
z	–	outer surface of the cylinder
$+\delta\xi^*$	–	with inaccurate location of the thermocouple by $\delta\xi^*$ closer to the edge
$-\delta\xi^*$	–	with inaccurate location of the thermocouple by $\delta\xi^*$ further from the edge
$\delta\vartheta$	–	with the measurement error equal to $\delta\vartheta$

Superscripts

$*$	–	measuring, Hermitian conjugate
T	–	transpose of a matrix

1 Introduction

Processes of heating elements of energy machines or processes of heat treatment require providing a thermal field satisfying the assumed criteria. To control the body heating it is important to know the temperature on the edge of the region. Not always is it possible to measure the temperature of the edge, for instance in a combustion chamber or on the inner surface of the body of a thermal turbine, and particularly it is difficult when the radiation is a great part of the heating process (process of heat treatment). In such cases the temperature of the edge could be determined by solving the inverse problem on the basis of temperature measurements at inner points in the body located close to its edge, on which the course of temperature is not known. Some methods of solving one-dimensional inverse problems of thermal fields distribution for a cylinder are presented in [1], and for a cylindrical layer – in [2]. Solving the inverse problem using the Laplace transform is presented in [1–3]. Solving the inverse heat conduction problem employing the sequential method is described in[4,5], and the analysis of thermal fields during unsteady heat transfer for irregular geometry was described in paper [6]. To stabilize the solution to the inverse problem different regularization methods are applied; they are described in papers [7–9]. The method of the inverse problem is used in the thermal stress analysis of pipelines [10] and the work of the heat exchangers [11]. In paper, employing the Laplace transform, the solution in a convolution form have been obtained comprising the unknown course of temperature of the edge which had been searched for on the basis of temperature measurements inside the body. Integration of the convolution was made assuming the distribution of temperature, T, on the integration interval $\langle t_i, t_i + \Delta t \rangle$ in the form $T(x,t) = T(x,t_i)\,\Theta + T(x,t_i+\Delta t)(1-\Theta)$, $\Theta \in (0,1)$, and the choice of parameter Θ was made by examining singular values of the matrix of the inverse problem and the sensitivity of the solution to the inverse problem to disturbances.

2 Direct problem

A basis for solving the inverse problem is obtaining a solution to the direct problem with the unknown boundary condition in a parametric form. The linear heat conduction equation for the symmetric thermal field in the cylinder may be written in the following form [12]:

$$\rho c \frac{\partial T}{\partial t} = \lambda \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) , \quad r \in (0, r_z) \ , \quad t > 0 \tag{1}$$

where r_z is the radius of the outer surface of the cylinder, with the following conditions
– initial condition

$$T(r, t = 0) = T_0 = const \ , \tag{2}$$

– boundary condition

$$T(r = r_z, t) = T_z(t) \ , \tag{3}$$

– condition of the infinite limit of the solution at point $r = 0$

$$|T(r = 0, t)| < \infty \ . \tag{4}$$

Dependences (1)–(6) were reduced to the nondimensional form by the following substitutions:

$$\xi = \frac{r}{r_z} \ , \quad \vartheta = \frac{T - T_0}{T_{\max}} \ , \quad \tau = \frac{\lambda}{\rho c} \frac{t}{r_z^2} \ , \tag{5}$$

where T_{max} is the maximum temperature while heating, and λ, ρ, c are the heat conduction coefficient of the cylinder, the density of the cylinder and the specific heat of the cylinder, respectively:
– differential equation (1)

$$\frac{\partial \vartheta}{\partial \tau} = \frac{\partial^2 \vartheta}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial \vartheta}{\partial \xi} \ , \quad \xi \in (0, 1) \ , \quad \tau > 0 \ , \tag{6}$$

– initial condition (4)

$$\vartheta(\xi, \tau = 0) = 0 \ , \tag{7}$$

– boundary condition (5)

$$\vartheta(\xi = 1, \tau) = \vartheta_z(\tau) \quad \tau > 0 \ , \tag{8}$$

– condition of the infinite limit of the solution (6) at point $\xi = 0$,

$$|\vartheta(\xi = 0, \tau)| < \infty \ . \tag{9}$$

To solve the problem (8)–(11) the Laplace transformation expressed by the dependence [12,13] was applied:

$$L\big[\vartheta\,(\xi,\tau)\big] = \bar{\vartheta}\,(\xi,\tau) = \int\limits_0^\infty \vartheta\,(\xi,\tau)\, e^{-s\tau} d\tau \ , \tag{10}$$

where s is a complex number. Having applied the Laplace transformation to the Eq. (8) and then employed the condition (10) it was obtained

$$\frac{d^2\bar{\vartheta}}{d\xi^2} + \frac{1}{\xi}\frac{d\bar{\vartheta}}{d\xi} = s\bar{\vartheta} - \vartheta\,(\xi,0) \tag{11}$$

and the general integral, assuming that $\vartheta\,(\xi,0) = \mathit{const}$, and taking the condition of infinite limit of the solution (11) into consideration is as follows [12–14]:

$$\bar{\vartheta}\,(\xi,s)\big|_{\xi=1} = \frac{1}{s}\left(s\bar{\vartheta}_z\,(s)\right) = C\; I_0\left(\sqrt{s}\xi\right)\big|_{\xi=1} - \frac{1}{s}\vartheta_0 \ . \tag{12}$$

Therefore,

$$C = \frac{1}{s}\frac{s\bar{\vartheta}_z\,(s) + \vartheta_0}{I_0\left(\sqrt{s}\right)} \ . \tag{13}$$

Then substituting the dependence (13) into the equation $\bar{\vartheta}\,(\xi,s) = CI_0 \times (\sqrt{s}\xi) - \frac{1}{s}\vartheta_0$ we have

$$\bar{\vartheta}\,(\xi,s) = \frac{I_0\left(\sqrt{s}\xi\right)}{sI_0\left(\sqrt{s}\right)}\left(s\bar{\vartheta}_z\,(s)\right) \ . \tag{14}$$

Therefore, employing the inverse Laplace transform and the Borel's convolution theorem [13] we obtain

$$\vartheta\,(\xi,\tau) = L^{-1}\left[\frac{I_0\left(\sqrt{s}\xi\right)}{sI_0\left(\sqrt{s}\right)}\left(s\bar{\vartheta}\,(\xi=1,s)\right)\right] = \tag{15}$$

$$= L^{-1}\left[\frac{I_0\left(\sqrt{s}\xi\right)}{sI_0\left(\sqrt{s}\right)}\right] * L^{-1}\big[s\bar{\vartheta}\,(\xi=1,s)\big] = L^{-1}\bar{\vartheta}_w\,(\xi,s) * L^{-1}\big[s\bar{\vartheta}\,(\xi=1,s)\big] \ ,$$

where $L^{-1}\left[s\bar{\vartheta}_z\,(s)\right] = \delta\,(\tau)\,\vartheta_z\,(0) + \vartheta_z'\,(\tau) = \vartheta_z'\,(\tau)$, and δ, the prime (') symbol, $*$ denote the Dirac delta function, differentation, convolution of functions, respectively. Employing the residue theorem [12] we determine the solution $\vartheta_w(\xi,\tau)$ being the solution to the problem with a constant temperature on the edge

$$\vartheta_w(\xi,\tau) = L^{-1}\left[\frac{I_0(\sqrt{s}\xi)}{sI_0(\sqrt{s})}\right] = \sum_{n=0}^{\infty} \operatorname*{res}_{s=s_n}\left[\frac{I_0(\sqrt{s}\xi)}{sI_0(\sqrt{s})}e^{s\tau}\right], \tag{16}$$

where s_n are poles of the function $\frac{I_0(\sqrt{s}\xi)}{sI_0(\sqrt{s})}e^{s\tau}$, i.e., numbers for whom the dominator $sI_0(\sqrt{s})$ is equal to zero. They are $s_0 = 0$ and roots of the equation $I_0(\sqrt{s}) = 0$. For $s_0 = 0$ we have

$$\operatorname*{res}_{s_0=0}\left[\frac{I_0(\sqrt{s}\xi)}{sI_0(\sqrt{s})}e^{s\tau}\right] = \lim_{s\to 0}\left[s\frac{I_0(\sqrt{s}\xi)}{sI_0(\sqrt{s})}e^{s\tau}\right] = \lim_{s\to 0}\left[s\frac{I_0(\sqrt{s}\xi)}{sI_0(\sqrt{s})}e^{s\tau}\right] = 1, \tag{17}$$

while for the remaining poles

$$\vartheta_w(\xi,\tau) = 1+\sum_{n=1}^{\infty} \operatorname*{res}_{s=s_n}\left[\frac{I_0(\sqrt{s}\xi)}{sI_0(\sqrt{s})}e^{s\tau}\right] = 1+\sum_{n=1}^{\infty} \lim_{s\to s_n}\left[(s-s_n)\frac{I_0(\sqrt{s}\xi)}{sI_0(\sqrt{s})}e^{s\tau}\right]. \tag{18}$$

Having employed l'Hôpital's rule and the properties for the modified Bessel functions, the dependence (18) takes the following form:

$$\vartheta_w(\xi,\tau) = 1+\sum_{n=1}^{\infty} \lim_{s\to s_n}\left[\frac{I_1(\sqrt{s}\xi)(s-s_n)e^{s\tau}\frac{1}{2\sqrt{s}\xi}+I_0(\sqrt{s}\xi)\left(e^{s\tau}+e^{s\tau}(s-s_n)\tau\right)}{I_0(\sqrt{s})+sI_1(\sqrt{s})\frac{1}{2\sqrt{s}}}\right]. \tag{19}$$

Since

$$\lim_{s\to s_n} I_1\left(\sqrt{s}\xi\right)(s-s_n)e^{s\tau}\frac{1}{2\sqrt{s}\xi} = 0 \tag{20}$$

and

$$\lim_{s\to s_n} I_0\left(\sqrt{s}\xi\right)(s-s_n)e^{s\tau} = 0, \tag{21}$$

therefore

$$\vartheta_w(\xi,\tau) = 1+\sum_{n=1}^{\infty} \lim_{s\to s_n}\left[\frac{2I_0(\sqrt{s}\xi)e^{s\tau}}{2I_0(\sqrt{s})+\sqrt{s}I_1(\sqrt{s})}\right]. \tag{22}$$

Numbers s_n are poles of the function $\frac{I_0(\sqrt{s}\xi)}{sI_0(\sqrt{s})}e^{s\tau}$ hence $\lim\limits_{s\to s_n} I_0(\sqrt{s}) = 0$ and

$$\vartheta_w(\xi,\tau) = 1+\sum_{n=1}^{\infty} \lim_{s\to s_n}\left[2\frac{I_0(\sqrt{s}\xi)e^{s\tau}}{\sqrt{s}I_1(\sqrt{s})}\right] = 1+2\sum_{n=1}^{\infty}\frac{I_0\left(\sqrt{s_n}\xi\right)e^{s_n\tau}}{\sqrt{s_n}I_1\left(\sqrt{s_n}\right)}. \tag{23}$$

The function I_0 of the real argument is positive, and for the imaginary argument is a function oscillating relative to the axis of abscissas, thus $\sqrt{s_n} = p_n i$, where i is the imaginary unit, then $s_n = -p_n^2$. Hence, the expression (19) takes the following form:

$$\vartheta_w(\xi,\tau) = 1 + 2\sum_{n=1}^{\infty} \frac{I_0(p_n i\xi)\, e^{-p_n^2\tau}}{p_n i I_1(p_n i)}\,. \tag{24}$$

Since $I_{-n}(s) = i^{-n} J_n(si)$ then $I_0(p_n i\xi) = J_0(p_n\xi)$ and $I_1(p_n i) = iJ_1(p_n)$ [12] hence

$$\begin{aligned}\vartheta_w(\xi,\tau) &= 1 + 2\sum_{n=1}^{\infty} \frac{J_0(p_n\xi)\, e^{-p_n^2\tau}}{p_n i i J_1(p_n)} = \\ &= 1 + 2\sum_{n=1}^{\infty} \frac{J_0(p_n\xi)\, e^{-p_n^2\tau}}{-p_n J_1(p_n)} = 1 - 2\sum_{n=1}^{\infty} \frac{J_0(p_n\xi)\, e^{-p_n^2\tau}}{p_n J_1(p_n)}\,.\end{aligned} \tag{25}$$

Numbers s_n are roots of the equation $I_0(\sqrt{s}) = 0 \Leftrightarrow I_0(pi) = J_0(p) = 0$. Hence, p_n are roots of the equation $J_0(p) = 0$. Therefore, the symmetric thermal field in the cylinder is expressed by the formula

$$\vartheta(\xi,\tau) = \frac{\partial\vartheta(\xi{=}1,\tau)}{\partial\tau} * \left[1 - 2\sum_{n=1}^{\infty} \frac{J_0(p_n\xi)\, e^{-p_n^2\tau}}{p_n J_1(p_n)}\right] = \frac{\partial\vartheta(\xi{=}1,\tau)}{\partial\tau} * \vartheta_w(\xi,\tau)\,. \tag{26}$$

The function $\vartheta_w(\xi,\tau)$ is the solution to the Eq. (8) with the initial condition (9) and constant temperature on the edge $\vartheta_w(\xi = 1,\tau) = 1$ (condition (10), $\vartheta_z = 1$).

3 Inverse problem

Since it is not always possible to determine temperature on the boundary of the region, such determination can be done by solving the inverse problem. In order to do that one should measure temperature at inner points of the region, Fig. 1. In the next step, the value of the function $\vartheta(\xi = 1,\tau)$ from the minimization of the distance between the temperature expressed by the formula (26) and the value measured for subsequent moments of time is searched for (for the case of a single measurement point, the distance between the temperature (26) and temperature measured becomes

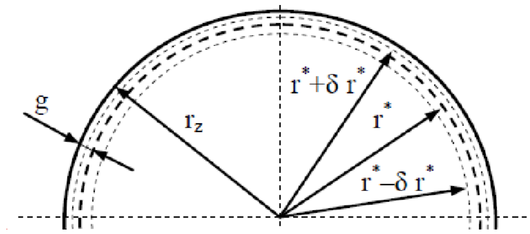

Figure 1: Figure of a cylinder cross-section with the outer radius (r_z), the point of thermocouple installation (r*,g) and the accuracy of the thermocouple location (r* + δ r*, r* – δr*).

the equality). The thermocouple is located at the distance of $g = r_z - r^*$ from the edge of the cylinder, hence $\xi^* = \frac{r^*}{r_z} = \frac{r_z - g}{r_z}$.

Convolution solution for the point of the temperature measurement $r = r^*$ $(\xi = \xi^*)$ is equal to the measured value, therefore, for two subsequent moments of time τ_i, τ_{i+1}we have:

$$\begin{cases} \vartheta\left(\xi^*,\tau_i\right) = \int\limits_0^{\tau_i} \vartheta'\left(\xi=1,u\right)\vartheta_w\left(\xi^*,\tau_i-u\right)du \ , \\ \vartheta\left(\xi^*,\tau_{i+1}\right) = \int\limits_0^{\tau_{i+1}} \vartheta'\left(\xi=1,u\right)\vartheta_w\left(\xi^*,\tau_{i+1}-u\right)du \ . \end{cases} \tag{27}$$

Having subtracted from both sides we obtain

$$I_2 = \vartheta\left(\xi^*,\tau_{i+1}\right) - \vartheta\left(\xi^*,\tau_i\right) = \int\limits_0^{\tau_{i+1}} \vartheta'\left(\xi=1,u\right)\vartheta_w\left(\xi^*,\tau_{i+1}-u\right)du - \int\limits_0^{\tau_i} \vartheta'\left(\xi=1,u\right)\vartheta_w\left(\xi^*,\tau_i-u\right)du \ . \tag{28}$$

Hence, substituting the integral on the interval $\langle 0,\tau_{i+1}\rangle$ by a sum of integrals over subsequent intervals $\langle \tau_n,\tau_{n+1}\rangle$, $n = 0,1,\ldots,i$

$$I_2 = \sum_{n=0}^{i} \int\limits_{\tau_n}^{\tau_{n+1}} \vartheta'\left(\xi=1,u\right)\vartheta_w\left(\xi^*,\tau_{i+1}-u\right)du - \sum_{n=0}^{i-1} \int\limits_{\tau_n}^{\tau_{n+1}} \vartheta'\left(\xi=1,u\right)\vartheta_w\left(\xi^*,\tau_i-u\right)du \ . \tag{29}$$

Since the function $\vartheta_w(\xi^*, \tau_i - u) > 0$ thus to simplify the convolution integration we approximate it by the staircase function $\vartheta_w(\xi^*, \tau_i - \tau_n)\,\Theta + \vartheta_w(\xi^*, \tau_i - \tau_{n+1})\,(1-\Theta)$, $\Theta \in (0,1)$ [15], Fig. 2. Hence, approximating the

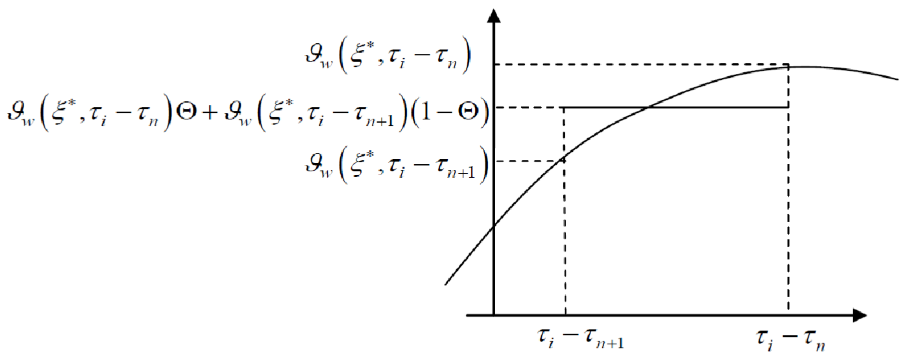

Figure 2: Approximation of the function ϑ_w by the staircase function.

functions $\vartheta_w(\xi^*, \tau_i - u)$ and $\vartheta_w(\xi^*, \tau_{i+1} - u)$, $u \in \langle \tau_n, \tau_{n+1} \rangle$ by the staircase function (Fig. 2) with the parameter $\Theta \in (0,1)$ the dependence (29) is expressed by the formula

$$\begin{aligned} I_2 = &\sum_{n=0}^{i} \int_{\tau_n}^{\tau_{n+1}} \vartheta'(\xi=1,u)\left[\vartheta_w(\xi^*, \tau_{i+1}-\tau_n)\,\Theta + \vartheta_w(\xi^*, \tau_{i+1}-\tau_{n+1})\,(1-\Theta)\right] du \\ &- \sum_{n=0}^{i-1} \int_{\tau_n}^{\tau_{n+1}} \vartheta'(\xi=1,u)\left[\vartheta_w(\xi^*, \tau_i-\tau_n)\,\Theta + \vartheta_w(\xi^*, \tau_i-\tau_{n+1})\,(1-\Theta)\right] du, \\ &0<\Theta<1 \end{aligned} \tag{30}$$

As a result of the approximation of the function $\vartheta_w(\xi, \tau)$ by the staircase function, integrands in the expression (30) are constant. Therefore

$$\begin{aligned} I_2 = &\sum_{n=0}^{i} \left[\vartheta_w(\xi^*, \tau_{i+1}-\tau_n)\,\Theta + \vartheta_w(\xi^*, \tau_{i+1}-\tau_{n+1})\,(1-\Theta)\right] \int_{\tau_n}^{\tau_{n+1}} \vartheta'(\xi=1,u)\,du \\ &- \sum_{n=0}^{i-1} \left[\vartheta_w(\xi^*, \tau_i-\tau_n)\,\Theta + \vartheta_w(\xi^*, \tau_i-\tau_{n+1})\,(1-\Theta)\right] \int_{\tau_n}^{\tau_{n+1}} \vartheta'(\xi=1,u)\,du\,. \end{aligned} \tag{31}$$

After integrating the derivative $\vartheta'_z(\xi=1,\tau)$ of the function of temperature distribution on the edge of the cylinder we obtained:

$$
\begin{aligned}
I_2 = & \sum_{n=0}^{i} \left[\vartheta_w\left(\xi^*, \tau_{i+1}-\tau_n\right)\Theta + \vartheta_w\left(\xi^*, \tau_{i+1}-\tau_{n+1}\right)(1-\Theta)\right]\times \\
& \left[\vartheta\left(\xi=1, \tau_{n+1}\right)-\vartheta\left(\xi=1, \tau_n\right)\right]\left(\tau_{n+1}-\tau_n\right)- \\
& -\sum_{n=0}^{i-1} \left[\vartheta_w\left(\xi^*, \tau_i-\tau_n\right)\Theta + \vartheta_w\left(\xi^*, \tau_i-\tau_{n+1}\right)(1-\Theta)\right]\times \\
& \left[\vartheta\left(\xi=1, \tau_{n+1}\right)-\vartheta\left(\xi=1, \tau_n\right)\right]\left(\tau_{n+1}-\tau_n\right)
\end{aligned}
\tag{32}
$$

Excluding the last term ($n = i$) from the first sum on the right side, we obtain

$$
\begin{aligned}
I_2 = & \left[\vartheta_w\left(\xi^*, \tau_{i+1}-\tau_i\right)\Theta + \vartheta_w\left(\xi^*, \tau_{i+1}-\tau_{i+1}\right)(1-\Theta)\right]\times \\
& \left[\vartheta\left(\xi=1, \tau_{i+1}\right)-\vartheta\left(\xi=1, \tau_i\right)\right]\left(\tau_{i+1}-\tau_i\right)+ \\
& +\sum_{n=0}^{i-1} \left\{\vartheta_w\left(\xi^*, \tau_{i+1}-\tau_n\right)\Theta + \vartheta_w\left(\xi^*, \tau_{i+1}-\tau_{n+1}\right)(1-\Theta)-\right. \\
& \left.-\left[\vartheta_w\left(\xi^*, \tau_i-\tau_n\right)\Theta + \vartheta_w\left(\xi^*, \tau_i-\tau_{n+1}\right)(1-\Theta)\right]\right\}\times \\
& \left[\vartheta\left(\xi=1, \tau_{n+1}\right)-\vartheta\left(\xi=1, \tau_n\right)\right]\left(\tau_{n+1}-\tau_n\right)
\end{aligned}
\tag{33}
$$

and introducing notations

$$
a_{i+1,i+1}=\left[\vartheta_w\left(\xi^*, \tau_{i+1}-\tau_i\right)\Theta+\vartheta_w\left(\xi^*, \tau_{i+1}-\tau_{i+1}\right)(1-\Theta)\right]\left(\tau_{i+1}-\tau_i\right) \tag{34}
$$

$$
\chi_{i+1}=\vartheta\left(\xi=1, \tau_{i+1}\right)-\vartheta\left(\xi=1, \tau_i\right), \quad i=0,1,2,\dots\ , \tag{35}
$$

$$
\begin{aligned}
a_{i,n} = & \left\{\vartheta_w\left(\xi^*, \tau_{i+1}-\tau_n\right)\Theta + \vartheta_w\left(\xi^*, \tau_{i+1}-\tau_{n+1}\right)(1-\Theta)-\right. \\
& \left.-\left[\vartheta_w\left(\xi^*, \tau_i-\tau_n\right)\Theta + \vartheta_w\left(\xi^*, \tau_i-\tau_{n+1}\right)(1-\Theta)\right]\right\}\left(\tau_{n+1}-\tau_n\right), \\
& i \geq 1, \quad n=0,1,\dots,i-1\ .
\end{aligned}
\tag{36}
$$

we obtain for $i > 1$ a system of equations with a triangular matrix

$$
\begin{bmatrix}
a_{11} & 0 & \dots & \dots & 0 \\
a_{21} & a_{22} & \ddots & & \vdots \\
a_{31} & a_{32} & a_{33} & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 0 \\
a_{i1} & a_{i2} & a_{i3} & \dots & a_{ii}
\end{bmatrix}
\begin{Bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \vdots \\ \chi_i \end{Bmatrix}
=
\begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_i \end{Bmatrix},
\tag{37}
$$

where

$$
B_j=\frac{\vartheta\left(\xi^*, \tau_{j+1}\right)-\vartheta\left(\xi^*, \tau_j\right)}{\tau_{j+1}-\tau_j}, \quad j=0,1,\dots,i-1\ , \tag{38}
$$

or in a compact form (for the constant time step $\Delta\tau = \tau_{i+1} - \tau_i$)

$$\left[A\left(\Theta, \vartheta_w\left(\xi^*\right), \Delta\tau\right)\right]\{\chi\} = \left\{B\left(\vartheta\left(\xi^*\right), \Delta\tau\right)\right\} \tag{39}$$

therefore

$$\{\chi\} = \left[A\left(\Theta, \vartheta_w\left(\xi^*\right), \Delta\tau\right)\right]^{-1}\left\{B\left(\vartheta\left(\xi^*\right), \Delta\tau\right)\right\} . \tag{40}$$

4 Sensitivity of solution

Each matrix equation in a form $A\chi = B$ may be characterized by its sensitivity. The sensitivity of the solution describes how the disturbance of data defining the problem, i.e., the matrix A and the vector B, influences the value of the solution χ. In case of the problem under consideration the sensitivity should be understood as the influence of the accuracy of the thermocouple location and of the temperature measurement on the error in calculated temperature distribution on the edge of the cylinder [16]. In calculations the accuracy of the thermocouple location was taken into account (the matrix A is treated as nondisturbed). It was assumed that the thermocouple may be located at points $\xi = \xi^* \pm \delta\xi^*$, where $\delta\xi^* = \frac{\delta r^*}{r_z}$. The temperature measurement is subject to a random error, being the time function and noted as $\delta\vartheta\left(\xi^*, \tau_i\right)$. This function takes the values $\delta\vartheta\left(\xi^*, \tau_i\right) \in \left\langle -\frac{\delta T^*}{T_{\max}}, \frac{\delta T^*}{T_{\max}} \right\rangle$, where δT^* and T_{max} are the absolute error of temperature at the measuring point and the maximum temperature while heating, respectively. On the basis of the formulae (37)–(40) the distribution of temperature on the edge of cylinder was determined with the presence of the installation error and the stochastic error of temperature measurement:

$$\left[A\left(\Theta, \vartheta_w\left(\xi^*\right), \Delta\tau\right)\right]\left\{\chi + \delta\chi\left(\delta\xi^*, \delta\vartheta, \tau\right)\right\} = \left\{B\left(\vartheta\left(\xi^*\right), \tau\right) + \delta B\left(\delta\xi^*, \delta\vartheta, \tau\right)\right\} \tag{41}$$

Then, subtracting (39) from (41) we have that

$$\left[A\left(\Theta, \vartheta_w\left(\xi^*\right), \Delta\tau\right)\right]\left\{\delta\chi\left(\delta\xi^*, \delta\vartheta, \tau\right)\right\} = \left\{\delta B\left(\delta\xi^*, \delta\vartheta, \tau\right)\right\} . \tag{42}$$

Therefore,

$$\left\{\delta\chi\left(\delta\xi^*, \delta\vartheta, \tau\right)\right\} = \left[A\left(\Theta, \vartheta_w\left(\xi^*\right), \Delta\tau\right)\right]^{-1}\left\{\delta B\left(\delta\xi^*, \delta\vartheta, \tau\right)\right\} , \tag{43}$$

where, on the basis of (38), we have

$$\delta B_j\left(\delta\xi^*,\delta\vartheta\right)= \frac{\vartheta\left(\xi^*\pm\delta\xi^*,\tau_{j+1}\right)+\delta\vartheta\left(\xi^*,\tau_{j+1}\right)-\left(\vartheta\left(\xi^*\pm\delta\xi^*,\tau_j\right)+\delta\vartheta\left(\xi^*,\tau_j\right)\right)}{\tau_{j+1}-\tau_j} -\frac{\vartheta\left(\xi^*,\tau_{j+1}\right)-\vartheta\left(\xi^*,\tau_j\right)}{\tau_{j+1}-\tau_j},\qquad j=0,1,\ldots,i-1\ . \tag{44}$$

5 Sensitivity of the calculation process by employing singular value decomposition

In order to examine the sensitivity of the solution $\{\chi\}$ the singular value decomposition (SVD) for the matrix A was applied; it allows the matrix A to be presented in the form of $U\Sigma V^*$, where U and V are unitary matrices, Σ is a diagonal matrix, and V^* denotes the conjugate transpose of V [17]. For the matrix A, whose elements are real numbers, the SVD decomposition allows the matrix A to be presented as $U\Sigma V^T$, where $UU^T = U^TU = VV^T = V^TV = I$. The subscript T on the matrices U and V denotes the matrix transpose of U or V, respectively. Therefore, from the matrix equation $A\chi = B$ we have

$$U\Sigma V^T\chi = B\ . \tag{45}$$

If the rank of the matrix A is $\mathrm{rank}A = r$, then the following dependences between the elements on the main diagonal of the matrix Σ occur:

$$\sigma_1\ge\sigma_2\ge\ldots\ge\sigma_r>\sigma_{r+1}=\ldots=\sigma_N=0\ . \tag{46}$$

Numbers σ_i are non-negative square roots of the matrix A^TA eigenvalues. The condition number of the function $\mathrm{cond}A$ defined as

$$\mathrm{cond}A=\frac{\sigma_1}{\sigma_r} \tag{47}$$

serves for the analysis on how the output values are sensitive to disturbances of the input data, and σ_r represents the smallest positive singular value of the matrix A. Columns of the matrix V are orthonormal eigenvectors of the matrix A^TA corresponding to those eigenvalues. Columns of the matrix U are orthonormal eigenvectors corresponding to eigenvalues of the matrix A^TA.

6 Numerical example

Sample calculations were conducted for the cylinder of the radius $r_z = 50$ mm, made of low-carbon steel of the following properties: density $\rho = 7841$ kg/m^3, specific heat $c = 456$ J/(kg K) and heat conduction coefficient $\lambda = 50.3$ W/(m K). To test the program, it was assumed that the non-dimensional distribution of temperature on the edge of the element is expressed as an exponential function $f(\tau) = 1 - e^{-\beta\tau}$, where β is a coefficient determining the rate of heating; for the sample under consideration it was assumed to be equal to 1.5. Time of heating the edge of the cylinder was $t = 680$ s. Assumed values correspond to the courses of heating elements in many industrial processes, for instance in a process of nitriding. It was assumed that the thermocouple is located at the distance of 2 mm from the edge of the cylinder, i.e., on the radius $r^* = r_z - 2$ mm= 48 mm. It was assumed that the accuracy of the thermocouple installation is equal to 0.5 mm, and the measurement is done with the accuracy of 2.2 °C. For the analysed sample, $\Theta = 0.5$ and $\Theta = 0.52$ were assumed.

Distributions of temperature on the element's edge, calculated employing the direct problem and the inverse problem with the random error of disturbance of the temperature measurement and with the error of the thermocouple installation are presented in Fig. 3. For $\Theta = 0.5$ the differences between the assumed temperature on the edge and the calculated one employing the method of inverse problem increase for subsequent time units and reach the values above 120 °C (Fig. 3a). A significant growth of the error of solution may already be noticed after 120 s from the moment of starting the heating process. With the growth of Θ from the value of 0.5 to 0.52 the deviation of values calculated emploing the inverse problem from the assumed function of temperature distribution on the edge of the cylinder decreases significantly, which corresponds to the decrease in the value of the condition number cond$A$. Therefore, a right choice of Θ influences significantly on the sensitivity of the solution.

The results of calculations of singular values for the matrix A, obtained employing the SVD decomposition for the time of heating the cylinder $t = 680$ s, $\beta = 1.5$; $r_z = 50$ mm and for the thermocouple located at the distance of $g = 2$ mm from the edge ($\xi^* = \frac{50-2}{50} = 0.96$) are presented in Tab. 1.

For $\Theta < 0.5$, the rank of matrix $A^T A$ is less than 680, then the matrix A is a singular one. For the value of $0.5 \leq \Theta \leq 1$, the rank of the matrix under consideration is equal to 680, and cond$A$ decreases from the value of 1310.501518 to 1.512672. High value σ_1 and cond$A$ for $\Theta \approx 0.5$

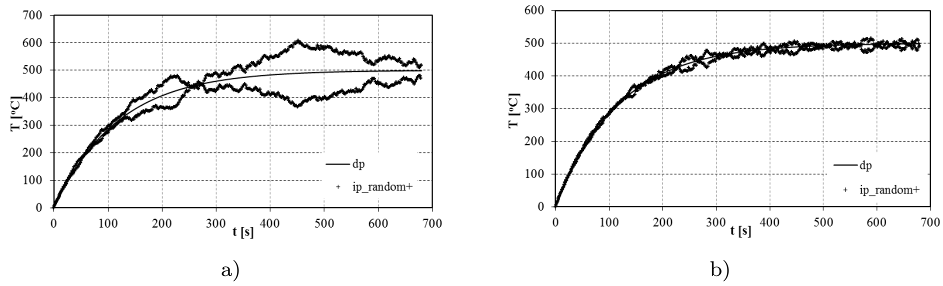

Figure 3: Distribution of temperature on the edge of the cylinder calculated on the basis of solving the direct problem (dp), the inverse problem with the error of the thermocouple location ($r = r^* + \delta r^*$) and random disturbance of temperature (ip_random+) for $\beta = 1.5$; $t = 680$ s; $r_z = 50$ mm, the thermocouple located $g = 2$ mm from the edge, and for parameter $\Theta = 0.5$ (a) and $\Theta = 0.52$ (b).

Table 1: Values σ_r of the minimum ($\sigma_r > 0$) and σ_1 of the maximum singular value for the matrix A, rank of the matrix A (rank$A$) and condition number cond$A$ for different values of Θ.

Θ	σ_1	σ_r	cond$A$	rank$A$
0, 0.1, 0.2	0	0	∞	0
0.3	∞	0	∞	547
0.4	∞	0	∞	544
0.5	1312.261387	1.001343	1310.501518	680
0.51	75.269699	1.001343	75.168758	680
0.52	37.806790	1.001343	37.756090	680
0.6	7.572991	1.001342	7.562839	680
0.7	3.786683	1.001342	3.781610	680
0.8	2.524479	1.001340	2.521100	680
0.9	1.893365	1.001339	1.890834	680
1	1.514694	1.001337	1.512672	680

($\Theta > 0.5$) causes the solution to be very sensitive to disturbances of the temperature measurement; whereas, with the growth of Θ the sensitivity of the temperature distribution on the edge of the cylinder to disturbances of the temperature measurement inside the cylinder decreases significantly.

7 Summary

This paper presents analytical and numerical solution to the inverse problem for unsteady heat exchange in the cylinder. The physical object was the cylinder of low-carbon steel of the diameter of 100 mm. While solving the direct problem it was assumed that the edge of the element was heated according to the function $f(\tau) = 1 - e^{-\beta\tau}$. It was assumed that the thermocouple is located at the distance of 2 mm from the edge and the accuracy of its location equals to ± 0.5 mm, and the temperature measurement is done with the accuracy of $\pm 2.2\,^{o}C$. The influence of the value of the integration parameter Θ on the sensitivity of the solution was considered. The sensitivity of the solution to the matrix equation $A\chi = B$ to an inaccurate location of the thermocouple and the stochastic error of the temperature measurement depends significantly on the value of Θ. With the occurrence of the random error of the temperature measurement for $\Theta = 0.5$ the maximum error of temperature, $\Delta T_{\max}$, determined on the edge of the element, reached the value above 120 °C; while for $\Theta = 0.52$ $\Delta T_{\max} \approx 25$ °C. On the basis of research conducted it can be concluded that a right choice of the value Θ during numerical integration may result in obtaining a solution that is significantly less sensitive to disturbances of the measurement data.

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