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Optimisation of the Railway Transition Curves' Shape with Use of Vehicle-track Dynamical Model

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Abstract

The concept and the method by present authors of searching for proper shape of transition curves are described in present paper. Short literature survey provides a background for such description. In the concept and method the advanced dynamical model of vehicle-track system, capable of simulation, and mathematically understood optimisation methods are exploited. Polynomial transition curve of any order can be optimised with the method proposed. Components of angular velocity and acceleration of transportation are presented for such curve in the paper. Thanks to them kinematical properties of the curve are represented in precise way in the dynamical model. As concerns the method, its principles and the most important details are discussed in the paper. Information about the method is extended by description of the software built to carry out the curve formation (optimisation) effectively. At the end, examples of the results generated by this software are presented and discussed.

Keywords: optimisation, numerical simulation, transition curves, railway vehicle dynamics, vehicle-track interactions

1. Introduction

The problem discussed in present article is the search for proper shape of railway transition curves. It is going to be done with regard to advanced vehicle model, dynamical track-vehicle and vehicle-passenger interactions, and optimisation

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methods. The search for proper shape means to the authors evaluation of the curve features based on chosen dynamical quantities as well as the generation of such shape with use of mathematically understood optimisation methods. The studies performed so far and those to be done in the future have got a character of the numerical tests. The key element for these tests is use of numerical simulation and the corresponding simulation software built by the authors. This software enables to simulate dynamics of the vehicle-track system exploiting complete dynamical model of the railway vehicle. Results of the simulations can be utilised in two general ways. First is the direct use to get the complete knowledge about the dynamical interactions and behaviour of the vehicle in transition curves. Second is the use of simulation results to determine the objective function in the process of optimising the shape of transition curve (TC). This second way of the results utilisation is discussed in the paper first of all.

1.1. Short literature survey

Review of the literature given below cannot be complete in terms of the cited and discussed publications since it would need a separate paper. Such a paper by present authors is [1]. Below we discuss the literature jointly, trying to focus on the features being the most common but not necessarily those most modern ones. Next, the criteria of TCs evaluation are classified, which are used most often.

In opinion of present authors the most general classification of the works on TCs might be their division into two groups. The first group includes those works that originate in the issues of railway road design (construction). So, these works deal with the railway infrastructure or in a wider view with the civil engineering. The second group is oriented towards railway vehicle dynamics. Thus strong relation to the mechanical engineering is clear. An increase in total number of works concerning TCs observed continuously [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] has not changed a proportion between number of the works in both groups. Despite the rise in number of works in the second group [11, 12, 13, 14, 15, 16, 17], the advantage in number of works in the first group is still a fact.

Recently, increase in number of publications that deal with transition curves, both the railway and the road ones, can also be observed. Also some qualitative change in their content can be noticed. It consists in attempts to diverge from the standard and to look for new, more modern methods of evaluating properties of TCs. Despite these, some earlier visible limitations of those works still exist, in present authors opinion. Namely, the analysis is rather rare which takes account of advanced dynamics of whole vehicle-track system. Present authors do not know method applied in practice (approved as a design tool), which use complete dynamical model of vehicle in formation of railway TCs. Many of the methods in use have got the traditional character. They are based on the traditional criteria (discussed below) and often refer to very simple vehicle model. The authors failed to find publications that exploit directly mathematically understood optimisation methods in formation of TCs, basing on objective functions calculated as a result of the numerical simulations. There are some works where selected quantities of interest, rather than the shape of the curve itself, are optimised instead.

In many works from the first group approach to the track-vehicle interactions is traditional, e.g. [18]. It is limited to discussing the vehicles jointly and studying the selected effects (quantities) in the car body. In such works the fundamental criteria of 3-dimensional TCs' formation are still crucial. Namely, the physical quantities that characterise effects on the passenger and eventually on the cargo should not exceed values that are acknowledged as acceptable [14, 18]. The corresponding relations refer to: unbalanced lateral acceleration $a \leqslant a_{lim}$, velocity of the a change $\psi \leqslant \psi_{lim}$, and velocity of wheel vertical rise along the superelevation ramp $f \leqslant f_{lim}$. Some up-to-date works extend these criteria with additional quantities and search for their courses. Such a quantity is the second derivative of a with respect to time t or distance l, in case of constant v. As to the courses (of the a first and second derivatives most often), the continuity (no abrupt change in values), differentiability (no bends) and so on are demanded. Despite that extension, such criteria do not take account of the dynamical properties of particular vehicle, including track-vehicle interactions in particular conditions, or effects on vehicle bogie. These are different than those adopted in the discussed criteria, where track has infinite stiffness and no geometrical irregularities, whereas vehicle is represented by a single rigid body or a particle.

Worthy of discussion are the criteria in the literature for evaluating TCs properties. Let us classify them into tree groups called: geometrical, geodetic, and dynamic criteria. In case of the geometrical criteria the appropriate boundary conditions at TC terminal points must be satisfied [14, 18, 19]. Also so called consistency conditions inside TC have to be satisfied. They ensure functions consistency (identity) of curvature k, superelevation ramp h, and unbalanced acceleration a. Apart from the acceptable boundary and maximum values also shape of the $(da/dt)=\psi$ and sometimes d^2a/dt^2 courses is of interest, e.g. [18].

The geodetic criteria compare different types of TCs from point of view of ease of their setting down and maintenance during operation, e.g. [11, 20].

In case of the dynamical criteria [11, 12, 13, 14, 15, 16, 17] dynamical response of the system on the input represented by TC (eventually including superelevation ramp) is of interest. Two approaches can be distinguished here. The first is to build relatively simple vehicle model that represents all vehicles and next to solve it analytically or numerically, e.g. [13, 16, 20]. The second is to build advanced vehicle or vehicle-track model and to solve it numerically (simulation) and then to analyse the results, e.g. [11, 12, 14, 15]. This type of criteria and works in type of the last mentioned confirm us in a conviction that the approach represented in our paper should be found justified and the results interesting.

2. Aims and Scope of the Article

2.1. The aim and scope in general

The main aim of the studies represented by present article is to elaborate and test the numerical method of TCs form optimisation. This is in agreement with the general concept of evaluating and forming TCs formulated in Sec. 1.

As to the scope, several introductory assumptions (expectations) have to be mentioned. First, pure mathematical methods of optimisation are going to be used. Second, dynamical quantities being the result of simulation of railway vehicle advanced (complete) model will be used to determine the objective functions in the optimisation process. Third, possibility of calculation a dozen or so different objective functions should be ensured in the software. Forth, search for optimum solutions with objective functions based on traditional criteria should be secured. Fifth, possibility to adopt all fundamental demands concerning TC should be preserved in the software. Sixth, two-axle vehicle model as well as vertically and laterally flexible track models are going to be used. Seventh, full non-linear geometry and forces in wheel/rail contact have to be taken into account. Eight, polynomial TC is going to be optimised. Ninth, the highest order of the polynomial should equal at least 12. Tenth, optimisation of the entrance and exit TCs as well as of both simultaneously, for the compound route, should be possible.

2.2. The object and the corresponding model

In order to make the analysis easier and more clear relatively simple object, and hence its relatively simple model were utilised. To make direct reference to the earlier results possible, the same model of the system was used as in the earlier studies by present authors [14]. Consequently, all simulation studies were done for discrete vehicle model of 2-axle freight car, as described in refs. [14, 21]. Its structure is shown in Fig. 1c. It is supplemented with discrete models of vertically and laterally flexible track shown in Figs. 1a and 1b, respectively. Parameters of the models are within average range of values for 2-axle freight cars and for track, too. Linearity of the vehicle suspension system was assumed. So, linear stiffness and damping elements in vehicle suspension were applied. The same concerned the track models. Here also linear stiffness and damping elements were utilised. One can find all parameters of the models in [14, 21].

Vehicle model is equipped with a pair of wheel/rail profiles that corresponds to the real ones. That is a pair of the nominal (i.e. unworn) S1002/UIC60 profiles that are used all over the Europe. Non-linear geometry of this pair is introduced into the model in a form of table with the contact parameters. In order to calculate non-linear tangential contact forces between wheel and rail well known FASTSIM program by J.J. Kalker was applied. Normal forces in the contact are not constant but influenced by both the geometry and the dynamical effects that make value of a wheelset vertical load variable.

Generalised approach to the modelling was used, as explained in [21, 22, 23]. Basically, dynamics of relative motion is used in that approach. This means that description of motion (dynamics) is relative to track-based moving reference frames. Dynamical equations of motion are equations of relative motion with terms depending on motion of the reference frames explicitly recorded. None of such terms is omitted in the equations. According to this method, the kinematic type non-linearities arising from rotational motions of bodies within our MBS model are taken into account, too. The term generalised refers fist of all to the generalised conditions of motion. So, the same generalised vehicle model describes vehicle dynamics in any conditions, i.e. in straight track (ST), circular curve (CC), and TC sections. The routes composed of such sections can also be analysed.

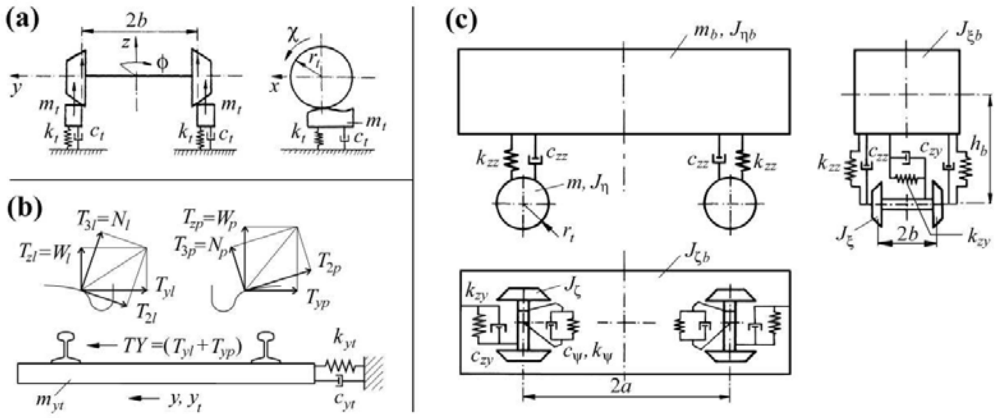

Fig. 1. System's nominal model: (a) track vertically, (b) track laterally, (c) vehicle

The route (section) of interest is characterised in the method by shape of the track centre line which is the general space (3-dimensional) curve. In railway systems such 3-dimensional objects are TCs with their superelevation ramps. A necessary condition to apply the method is description of the curves (sections) by parametric equations, with the curve's current length l as the parameter. The cases of CC and ST are treated in the method as the special cases of 2-dimensional and 1-dimensional geometrical objects, respectively. Such an approach was described in [19, 23].

An important element in the method is description of kinematics of the track-based moving reference frames. Their motion comes out directly from the track centre line shape. The applied method of determination of the kinematical quantities on the basis of the parametric equations is presented most recently in [23].

3. The Method for the Optimum Transition Curve Formation

In order to find the proper shape of the transition curve successfully, what is specified in Section 1 as the final aim of the undertaken activities, some intermediate (partial) aims have to be achieved. Generally, they concern building the method and the software for TC's shape optimisation and next their use and the analysis of the results obtained. The intermediate aims are as follows:

- determination of the initial transition curves type (shape) being a base in looking for the optimum shape,
- determination of at least dozen or so quality functions (*QF*s) being the criteria in the shape optimisation process,
- extension of the simulation software so that values of the *QF*s are determined (calculated) as a result of the simulation of vehicle motion,
- performing comparative simulation studies making possible to compare the results for different curves being the effect of the shape optimisation for the different criteria (*QF*s).

3.1. Type of the transition curve and demands for it

Type of a TC chosen for optimisation is the polynomial TC of any order $n \geq 4$. It is defined by Eqs. (1)-(4) that are related to space curve parametric equations:

$$y = \frac{1}{R}\left(\frac{A_n l^n}{L^{n-2}} + \frac{A_{n-1} l^{n-1}}{L^{n-3}} + \frac{A_{n-2} l^{n-2}}{L^{n-4}} + \frac{A_{n-3} l^{n-3}}{L^{n-5}} + ... + \frac{A_4 l^4}{L^2} + \frac{A_3 l^3}{L^1}\right) \tag{1}$$

$$k = \frac{d^2 y}{dl^2} = \frac{1}{R}\left[n(n-1)\frac{A_n l^{n-2}}{L^{n-2}} + (n-1)(n-2)\frac{A_{n-1} l^{n-3}}{L^{n-3}} + ... + 3\cdot 2\frac{A_3 l^1}{L^1}\right] \tag{2}$$

$$h = H\left[n(n-1)\frac{A_n l^{n-2}}{L^{n-2}} + (n-1)(n-2)\frac{A_{n-1} l^{n-3}}{L^{n-3}} + ... + 4\cdot 3\frac{A_4 l^2}{L^2} + 3\cdot 2\frac{A_3 l^1}{L^1}\right] \tag{3}$$

$$\begin{aligned} i = \frac{dh}{dl} = H\Bigg[&n(n-1)(n-2)\frac{A_n l^{n-3}}{L^{n-2}} + (n-1)(n-2)(n-3)\frac{A_{n-1} l^{n-4}}{L^{n-3}} \\ &+ ..... + 5\cdot 4\cdot 3\frac{A_5 l^2}{L^3} + 4\cdot 3\cdot 2\frac{A_4 l^1}{L^2} + 3\cdot 2\cdot 1\frac{A_3 l^0}{L^1}\Bigg] \end{aligned} \tag{4}$$

where y, k, h, and i define curve lateral co-ordinate, curvature, superelevation, and inclination of superelevation ramp, respectively. The R, H, L, and l define curve minimum radius (at its end), maximum susperelevation (at the curve end), total curve length, and curve current length, respectively. The A_i are polynomial coefficients ($i = n, n-1, \ldots, 4, 3$) while n is polynomial order. Number of the polynomial terms (terms in Eqs. (1)-(4)) must not be smaller than 2. On the other hand the smallest order $n_{\min}$ of the last term in Eq. (1) must be $n_{\min} \geq 3$. Such definition of the curves gives possibility of proper k and h values at TCs terminal points. They should equal 0 at the initial points and $1/R$ and H at the end points. Note, that values for both always equal 0 for l=0. In order to ensure $1/R$ and H values for $l = L$, normalisation of the coefficients is necessary. It is done with *WSR* coefficient defined in Eq. (5). The normalised coefficients A'_i are given in Eq. (6).

$$\begin{gathered} [n(n-1)A_n + (n-1)(n-2)A_{n-1} + \ldots.. + 4 \cdot 3 \cdot A_4 + 3 \cdot 2 \cdot A_3] = WSR \\ (1/WSR) \cdot [n(n-1)A_n + (n-1)(n-2)A_{n-1} + \ldots.. + 4 \cdot 3 \cdot A_4 + 3 \cdot 2 \cdot A_3] = 1 \end{gathered} \tag{5}$$

$$\begin{array}{ll} A'_n \;=\; A_n/WSR & \ldots\ldots.\ A'_5 = A_5/WSR \\ A'_{n-1} = A_{n-1}/WSR & A'_4 = A_4/WSR \\ A'_{n-2} = A_{n-2}/WSR\ \ldots\ldots\ldots. & A'_3 = A_3/WSR \end{array} \tag{6}$$

Finally, form of each TC being tested for optimum shape during the optimisation process is determined with the following parametric equations:

$$\begin{aligned} & x \cong l \\ & y = \frac{1}{R}\left(\frac{A'_n l^n}{L^{n-2}} + \frac{A'_{n-1} l^{n-1}}{L^{n-3}} + \frac{A'_{n-2} l^{n-2}}{L^{n-4}} + \frac{A'_{n-3} x^{n-3}}{L^{n-5}} + \ldots\ldots. + \frac{A'_5 l^5}{L^3} + \frac{A'_4 l^4}{L^2}\right) \\ & z = \frac{H}{2}\left[n(n-1)\frac{A'_n l^{n-2}}{L^{n-2}} + (n-1)(n-2)\frac{A'_{n-1} l^{n-3}}{L^{n-3}} + \ldots.. + 5 \cdot 4 \frac{A'_5 l^3}{L^3} + 4 \cdot 3 \frac{A'_4 l^2}{L^2}\right] \end{aligned} \tag{7}$$

In order to ensure tangence of the k and h functions at their terminal points to the k and h functions for ST and CC, values of i function given in Eq. (4) should equal 0 for l=0 and $l = L$. Equation (4) always produce 0 for l=0 when its last term is removed. So, this is the first condition that affects also rest of the equations. To ensure 0 value for $l = L$ the square brackets in (4) with the last term omitted should equal 0. In order to make it true all the coefficients remain unchanged but not just one arbitrarily selected. Its value is taken as opposite to the sum of those unchanged. And this is the second condition, that affects also rest of the equations. In case the second coefficient is selected for change it would be:

$$\begin{aligned} & \frac{1}{L}\left[n(n-1)(n-2)\frac{A'_n l^{n-3}}{L^{n-3}} + (n-1)(n-2)(n-3)\frac{A'_{n-1} l^{n-4}}{L^{n-4}} + \ldots.. + 4 \cdot 3 \cdot 2 \frac{A'_4 l^1}{L^1}\right] = 0 \\ & A''_{n-1} = -\frac{1}{(n-1)(n-2)(n-3)}\left[n(n-1)(n-2)A'_n + \ldots.. + 5 \cdot 4 \cdot 3 A'_5 + 4 \cdot 3 \cdot 2 A'_4\right] \end{aligned} \tag{8}$$

3.2. Kinematical properties of the polynomial transition curve of any order

In order to reflect precisely kinematical properties of the TC chosen in the previous subsection the components of angular velocity $\boldsymbol{\omega}$ and acceleration $\boldsymbol{\varepsilon}$ of transportation must be known. These are quantities that represent TC shape in the dynamical model. As mentioned earlier, the general method of determination of these components is presented in [23]. Fundamental relationships, invoked from [23], that define these components in the natural (moving trihedral) system are as follows:

$$\boldsymbol{\omega} = \boldsymbol{t} \cdot \omega_t + \boldsymbol{n} \cdot \omega_n + \boldsymbol{b} \cdot \omega_b \cong \boldsymbol{t} \cdot (\mathrm{d}\gamma/\mathrm{d}t) + \boldsymbol{b} \cdot vk \tag{9}$$

$$\begin{aligned} \boldsymbol{\varepsilon} &= \boldsymbol{t} \cdot \varepsilon_t + \boldsymbol{n} \cdot \varepsilon_n + \boldsymbol{b} \cdot \varepsilon_b = \boldsymbol{t} \cdot (\mathrm{d}\omega_t/\mathrm{d}t) + \boldsymbol{n} \cdot (vk\omega_t + v\tau\omega_b) + \boldsymbol{b} \cdot (\mathrm{d}\omega_b/\mathrm{d}t) \\ &= \boldsymbol{t} \cdot \left(\mathrm{d}^2\gamma\big/\mathrm{d}t^2\right) + \boldsymbol{n} \cdot \left[vk\,(\mathrm{d}\gamma/\mathrm{d}t) + v^2\tau k\right] + \boldsymbol{b} \cdot [\mathrm{d}\,(\mathrm{vk})/\mathrm{d}t] \end{aligned} \tag{10}$$

where $\boldsymbol{t}, \boldsymbol{n},$ and $\boldsymbol{b}$ are versors of the natural system axes, γ is angle corresponding to superelevation h, and v is vehicle (variable) speed. It is seen in Eqs. (9) and (10) that one must know γ, v, k, and τ as well as some derivatives of these quantities in order to calculate components ω_t, ω_n, ω_b and ε_t, ε_n, ε_b. It can also be expected, when looking at Eqs. (9) and (10), that full analytical form of the components will be the complex one. That is why we do not tend to present full analytical form of the ω_t, ω_n, ω_b and ε_t, ε_n, ε_b in this paper. Instead, we will present form of the factors and terms that are directly used by us while calculating values of the components in the numerical model (code of the software).

Let us start with the angle γ. According to [23] the following formula holds:

$$\gamma = arc\sin[z(l)/b] \cong z(l)/b \tag{11}$$

where b is a half of the track gauge. The approximate version of (11) holds for small values of the angle. Note, that in real track $\gamma \leq 6^\circ$. Consequently:

$$\begin{aligned} \gamma &\cong \frac{H}{2b}\left[n(n-1)\frac{A'_n l^{n-2}}{L^{n-2}} + (n-1)(n-2)\frac{A'_{n-1} l^{n-3}}{L^{n-3}} + \ldots.. + 5 \cdot 4\frac{A'_5 l^3}{L^3} + 4 \cdot 3\frac{A'_4 l^2}{L^2}\right] \\ \frac{\mathrm{d}\gamma}{\mathrm{d}t} &= \frac{\mathrm{d}\gamma}{\mathrm{d}l} \cdot \frac{\mathrm{d}l}{\mathrm{d}t} = v \cdot \frac{\mathrm{d}\gamma}{\mathrm{d}l} = \frac{v}{2b} \cdot \frac{\mathrm{d}h}{\mathrm{d}l} = \frac{v}{2b} \cdot i \end{aligned} \tag{12}$$

$$\begin{aligned} &= \frac{vH}{2b}\left[n(n-1)(n-2)\frac{A_n l^{n-3}}{L^{n-2}} + (n-1)(n-2)(n-3)\frac{A_{n-1} l^{n-4}}{L^{n-3}}\right. \\ &\qquad \left. + \ldots.. + 5 \cdot 4 \cdot 3\frac{A_5 l^2}{L^3} + 4 \cdot 3 \cdot 2\frac{A_4 l^1}{L^2} + 3 \cdot 2 \cdot 1\frac{A_3 l^0}{L^1}\right] \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2} &= \mathrm{d}\left(\frac{vi}{2b}\right)/\mathrm{d}t = \frac{1}{2b} \cdot \frac{\mathrm{d}\,(vi)}{\mathrm{d}t} = \frac{1}{2b}\left(\frac{\mathrm{d}v}{\mathrm{d}t} \cdot i + v \cdot \frac{\mathrm{d}i}{\mathrm{d}t}\right) \\ &= \frac{1}{2b}\left(a \cdot i + v \cdot \frac{\mathrm{d}l}{\mathrm{d}t} \cdot \frac{\mathrm{d}i}{\mathrm{d}l}\right) = \frac{1}{2b}\left(a \cdot i + v^2 \cdot \frac{\mathrm{d}i}{\mathrm{d}l}\right) \end{aligned} \tag{14}$$

where in Eq. (14) the acceleration $a = \mathrm{d}v/\mathrm{d}t$ can be assumed as known. It is like that because change of the v and l in time must be known in case we want to consider the relative kinematics. Let us recall that v is velocity of the transportation system origin. The i is defined by (4) and di/dl is given below.

$$\frac{\mathrm{d}i}{\mathrm{d}l} = H\left[n(n-1)(n-2)(n-3)\frac{A'_n l^{n-4}}{L^{n-2}} + (n-1)(n-2)(n-3)(n-4)\frac{A'_{n-1} l^{n-5}}{L^{n-3}} + ... + 5\cdot 4\cdot 3\cdot 2\frac{A'_5 l^1}{L^3} + 4\cdot 3\cdot 2\cdot 1\frac{A'_4 l^0}{L^2}\right] \tag{15}$$

Now, let us discuss curvature k that is also present in Eqs. (9) and (10). We have in fact two options of its calculation. The first is direct use of (2). It is a simplified formula for the k. It generates small errors in general. Discussion of such errors' value is done in [23]. The other option is use of the non-simplified formula that holds for any 3-dimensional curve represented by the parametric equations. It is as follows:

$$k = \sqrt{\left(\mathrm{d}^2x/\mathrm{d}l^2\right)^2 + \left(\mathrm{d}^2y/\mathrm{d}l^2\right)^2 + \left(\mathrm{d}^2z/\mathrm{d}l^2\right)^2} \tag{16}$$

It is seen from Eq. (7) that first of the terms under the square root sign equals 0. Using Eqs. (3), (4) and (7) the missed term for the z co-ordinate can be calculated as:

$$\frac{\mathrm{d}^2z}{\mathrm{d}l^2} = \frac{1}{2}\cdot\frac{\mathrm{d}^2h}{\mathrm{d}l^2} = \frac{1}{2}\cdot\frac{\mathrm{d}i}{\mathrm{d}l} \tag{17}$$

where di/dl is given in Eq. (15). In our further calculations and in the software used to generate the simulation and optimisation results the first option is used.

The next to discuss is torsion τ of the curve that is also necessary to determine the kinematical components from Eqs. (9) and (10). Let us start with the general formula for the torsion known in the differential geometry.

$$\tau = \frac{1}{k^2}\cdot\begin{vmatrix} \mathrm{d}x/\mathrm{d}l & \mathrm{d}y/\mathrm{d}l & \mathrm{d}z/\mathrm{d}l \\ \mathrm{d}^2x/\mathrm{d}l^2 & \mathrm{d}^2y/\mathrm{d}l^2 & \mathrm{d}^2z/\mathrm{d}l^2 \\ \mathrm{d}^3x/\mathrm{d}l^3 & \mathrm{d}^3y/\mathrm{d}l^3 & \mathrm{d}^3z/\mathrm{d}l^3 \end{vmatrix} \tag{18}$$

In order to avoid unnecessary calculation one can note that first column of the determinant in (18) equals 1, 0, and 0. Then (18) can be recorded as follows:

$$\begin{aligned} \tau &= \frac{1}{k^2}\cdot\begin{vmatrix} 1 & \mathrm{d}y/\mathrm{d}l & \mathrm{d}z/\mathrm{d}l \\ 0 & \mathrm{d}^2y/\mathrm{d}l^2 & \mathrm{d}^2z/\mathrm{d}l^2 \\ 0 & \mathrm{d}^3y/\mathrm{d}l^3 & \mathrm{d}^3z/\mathrm{d}l^3 \end{vmatrix} = ... \\ &= \frac{1}{k^2}\left[\left(\mathrm{d}^2y/\mathrm{d}l^2\right)\cdot\left(\mathrm{d}^3z/\mathrm{d}l^3\right) - \left(\mathrm{d}^2z/\mathrm{d}l^2\right)\cdot\left(\mathrm{d}^3y/\mathrm{d}l^3\right)\right] \\ &= \frac{1}{k^2}\left[k\cdot\left(\mathrm{d}^3z/\mathrm{d}l^3\right) - \frac{1}{2}\frac{\mathrm{d}i}{\mathrm{d}l}\cdot\left(\mathrm{d}^3y/\mathrm{d}l^3\right)\right] \end{aligned} \tag{19}$$

In order to make use of last line in Eq. (19), the two expressions in round brackets have to be calculated. Looking at Eqs. (15) and (17) one can note that

$$\frac{\mathrm{d}^3 z}{\mathrm{d}l^3} = \mathrm{d}\left(\frac{1}{2}\cdot\frac{\mathrm{d}i}{\mathrm{d}l}\right)/\mathrm{d}l = \frac{H}{2}\left[n(n-1)(n-2)(n-3)(n-4)\frac{A'_n l^{n-5}}{L^{n-2}}\right.$$
$$\left. +(n-1)(n-2)(n-3)(n-4)(n-5)\frac{A'_{n-1} l^{n-6}}{L^{n-3}} + \ldots\ldots + 5\cdot 4\cdot 3\cdot 2\cdot 1\frac{A'_5 l^0}{L^3}\right] \tag{20}$$

Taking account of Eqs. (2)-(4) one can write down that:

$$\frac{\mathrm{d}^3 y}{\mathrm{d}l^3} = \frac{1}{RH}\cdot i \tag{21}$$

where i is determined with Eq. (4).

The last term that need to be calculated while determining the components of angular velocity and acceleration of transformation is the last term in Eq. (10). It defines the ε_b component. One can perform the following manipulation for it:

$$\frac{\mathrm{d}(vk)}{\mathrm{d}t} = \left(\frac{\mathrm{d}v}{\mathrm{d}t}\cdot k + v\cdot\frac{\mathrm{d}k}{\mathrm{d}t}\right) = \left(a\cdot k + v\cdot\frac{\mathrm{d}l}{\mathrm{d}t}\cdot\frac{\mathrm{d}k}{\mathrm{d}l}\right)$$
$$= \left(a\cdot k + v^2\cdot\frac{\mathrm{d}k}{\mathrm{d}l}\right) = \left(a\cdot k + v^2\cdot\frac{\mathrm{d}^3 y}{\mathrm{d}l^3}\right) \tag{22}$$

where $\mathrm{d}^3y/\mathrm{d}l^3$ is given through Eq. (21).

This way the components ω_t, ω_n, ω_b and ε_t, ε_n, ε_b became determinate. The presented formulae are used in the numerical code we elaborated. Note, that so defined components concern the natural system. If one needs the components in the transportation system then they must be transformed. It is done with use of the direction cosine matrix between these co-ordinate systems. The matrix and transformation itself are presented in detail in [23].

3.3. Optimisation Method and Objective Functions

Let us formulate the optimisation problem that is going to be solved in the presented studies. So, one ought to find A'_i, being the polynomial coefficients, ensuring possibility of imposing constraints on their values. The constraints in the form $b_i < A'_i < c_i$ should enable to control form of the polynomials in case it is necessary. Namely, the forms of polynomials reasonable from practical point of view should be tested in succeeding steps of the optimisation process, rather than those being impossible to apply in a railway conditions. The problem just formulated is a classical formulation of the static constrained optimisation. It is realised with the library procedure that utilises moving penalty function algorithm combined with Powell's method of conjugate directions.

The difficulty of the problem solution lies in the very complex objective function (quality function QF). This function is calculated as a result of the numerical simulation of dynamical mechanical system (as described in Subsecs. 2.1 and 4.1). The main steps during calculation of the objective function are: generation of the new shape of TC, calculation of kinematical quantities (velocities and accelerations of transformation) that depend on this new shape, and solution of ODEs set.

At current stage of the studies nine quality functions (QF) were implemented in the software. They concern minimisation of: sum of lateral displacements for both wheelsets (QF_1); sum of squares of longitudinal and lateral creepages in wheel/rail contact for all wheels (QF_2); sum of yaw angles for both wheelsets (QF_3); wheel/rail wear represented by sum of products of tangential forces by creepages in longitudinal and lateral directions for all wheels (QF_4); wheel/rail wear represented by sum of products of normal forces and resultant creepages for all wheels (QF_5); sum of normal forces for all wheels (QF_6); sum of lateral accelerations of wheelsets and car body (QF_7); sum of lateral accelerations of wheelsets (QF_8); and lateral acceleration of car body (QF_9).

So far just three functions were tested. They are QF_2, QF_7, and QF_8. Their physical interpretation is as follows. Minimisation of QF_2 causes less wear of wheels and rails, of QF_7 reduces vehicle-track and vehicle-passenger interactions, while of QF_8 reduces vehicle-track interactions. These three QFs are given in Eqs. (23), (24), and (25):

$$QF_2 = L_C^{-1} \int_0^{L_C} \left(\nu_{xlp}^2 + \nu_{ylp}^2 + \nu_{xrp}^2 + \nu_{xrp}^2 + \nu_{xlk}^2 + \nu_{ylk}^2 + \nu_{xrk}^2 + \nu_{yrk}^2\right) dl \tag{23}$$

$$QF_7 = L_C^{-1} \int_0^{L_C} \left(|\ddot{y}_p| + |\ddot{y}_k| + |\ddot{y}_b|\right) dl \tag{24}$$

$$QF_8 = L_C^{-1} \int_0^{L_C} \left(|\ddot{y}_p| + |\ddot{y}_k|\right) dl \tag{25}$$

where L_C is distance where QF is calculated, ν_x is longitudinal creepage, ν_y is lateral creepage, and $\ddot{y}$ is lateral acceleration. Indices l, r, p, k, b refer to left hand-side, right hand-side, leading wheelset, trailing wheelset, and vehicle body, respectively.

Some problem in optimising form of TC is a starting point (initial form of TC). In case of TCs of 5th, 7th, and 9th orders used in the calculations so far, we utilised the curves known in the literature as the best ones from point of view of the traditional criteria. For example, in case of the 5th order it was the Bloss (Glodner) curve. Generally, such curves ensure minimum of the centrifugal force integral inside a TC. This is satisfied for the simplest possible model, however, that is in fact represented by a particle. For the higher order polynomial TCs form of the initial TC was an arbitrary choice, possibly in connection with TCs found in the literature. The curves in view are given below.

$$y = \frac{1}{R}\left(-\frac{l^5}{10L^3} + \frac{l^4}{4L^2}\right) \tag{26}$$

$$y = \frac{1}{R}\left(\frac{l^7}{7L^5} - \frac{l^6}{2L^4} + \frac{l^5}{2L^3}\right) \tag{27}$$

$$y = \frac{1}{R}\left(-\frac{5l^9}{18L^7} + \frac{5l^8}{4L^6} - \frac{2l^7}{L^5} + \frac{7l^6}{6L^4}\right) \tag{28}$$

4. Characteristics of the Software

The software implementing the method introduced in the previous section is composed of two main parts. The first is the software for a vehicle-track dynamics simulation. The second is the optimisation software. This part is represented by the library procedure. Both parts are represented by the Fortran numerical code. Both parts are united in the single software package. The basic tasks undertaken while building the package were to:

- make it possible to take any shape of the transition curve and any reasonable configuration of the test route for the simulation (within adopted range of the studies),
- make it possible to take any shape of the corresponding superelevation ramp,
- make it possible to calculate the imaginary forces (the inertia forces depending on the curve shape) for any shape of the transition curve proposed (selected) by the optimisation software (determining the components of the angular velocity and acceleration of transportation),
- combine into single software package the simulation software and the optimisation procedures.

4.1. The software for vehicle-track dynamics simulation

Simulation software reflects in numerical form all the features specified in subsection 2.1 for the mathematical model. So, the original as well as characteristic features of the model are represented in the software. Despite many such features, in general view our software might be defined as typical railway vehicle dynamics simulation software. We mean that the aims for its use and results generated are those typical.

The software in view was built by the authors on their own, so it is not a commercial code. It is also not software for automatic generation of equations of motion (AGEM) but the one corresponding to a particular mechanical system. Here it corresponds to the system defined in Fig. 1. It was developed, tested, and used in research for many years. The base in this software is set of the 2nd order ordinary differential equations (ODEs). These equations of motion were derived from Lagrange equations of type II adapted to description of relative motion. General form of such equations is presented in [19, 21]. Number of the equations equals a number of degrees of freedom (DOFs) of the system. Complete vehicle-track system possesses 18 DOFs. The equations of 2nd order are transformed within the code into the 1st order ODEs to integrate them. The integration is done with use of methods capable to solve stiff differential equations (Gear's methods). Direct results of the numerical integration are obviously the system co-ordinates and velocities. Apart from them, the broad set of other supplementary quantities can also be obtained. For example these could be forces and torques exploited in the simulations. The results can either be recorded in the resultant files or used directly in the calculations. Both the direct and the supplementary results are used while calculating values of objective functions during the optimisation.

4.2. General representation of the software for transition curves formation

Scheme of the software is shown in Fig. 2. Two iteration loops are visible there. The first is an integration loop. This loop is stopped when distance l_{lim} being length of the route (usually the compound route ST, TC, and CC) is reached by vehicle model. The second is an optimisation process loop. It is stopped when number of iterations reaches limit value i_{lim}. This value means that i_{lim} simulations have to be performed in order to stop optimisation process. In the calculations done so far i_{lim} equal 100 was used. If optimum solution is reached earlier, i.e. for $i < i_{lim}$ then the optimisation process stops automatically and the corresponding results are recorded. The post-processor ensures record of A_i' coefficients in each optimisation step in the special file. Besides results of simulations for the initial and optimum TC shapes are recorded automatically in the resultant files.

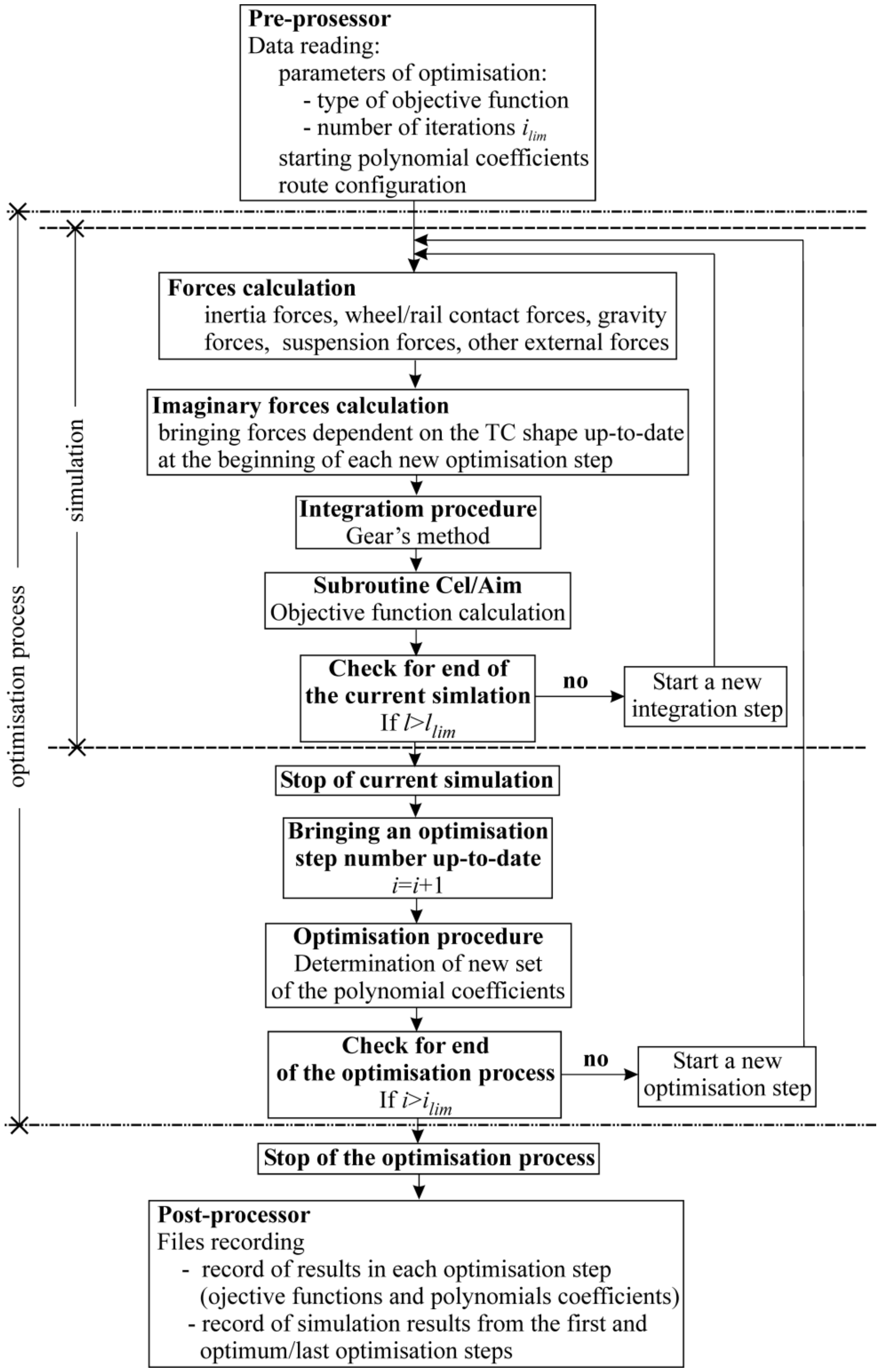

Fig. 2. General scheme of the optimisation software

5. Results of the Numerical Tests

The results to be presented are divided into two parts. The aim for the first part is demonstration that the method works well and that results of optimisations can be useful in the future analyses. The aim for the second part is to show what kind of results can be generated and how the first analyses look like. All the results below refer to the same compound route. Its data are: velocity v=45 m/s, (ST; L=50 m), (TC; L=102,5 m, $R_{\min}$=600 m, $H_{\max}$=0,16 m), (CC; L=100 m, R=600 m, H=0,16 m).

5.1. Results aimed at testing the software

It is seen in Figs. 3 and 4, for the 7th order TCs, that searching for the optimum TC is successful. Usually quite wide range of the shapes is swept (tested). It can also be seen that applied method of forcing the software to take proper values for k and h at TC's terminal points works right. By comparing these figures it can also be seen that the method ensuring tangence of k and h functions for TC at its terminal points to the functions for ST and CC works right, too. It can be observed for the shapes producing k and h values smaller than 0 or higher than $1/R$ and H that such values are trimmed off by the software to avoid model's "derailment". In case "derailment" persists, QFs are given very high values to stop interest of the software in such solutions. Coefficients A'_i for the optimised TCs are given in Sec. 6.

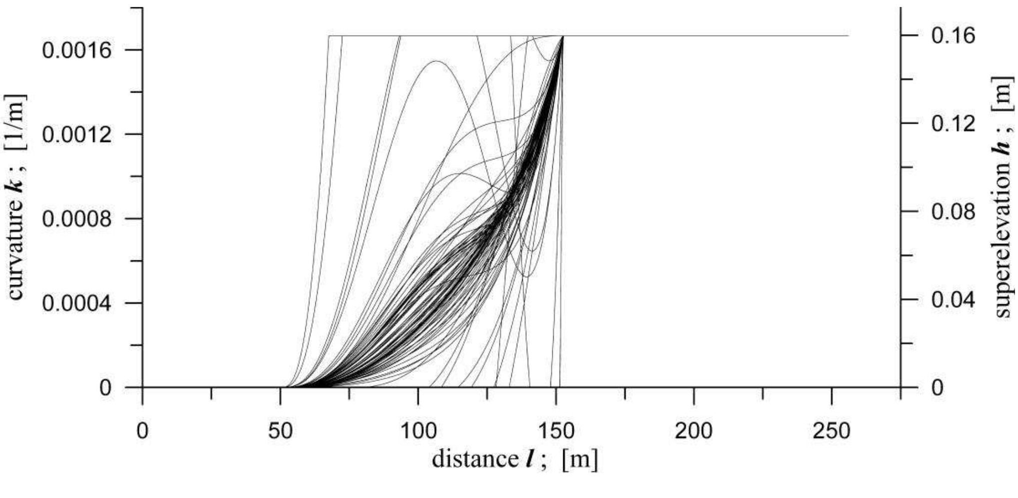

Fig. 3. Search for optimum 7th order TC's shape, represented by curvature k and superelevation h. No conditions for the tangence of the k and h is imposed. QF_2 is used

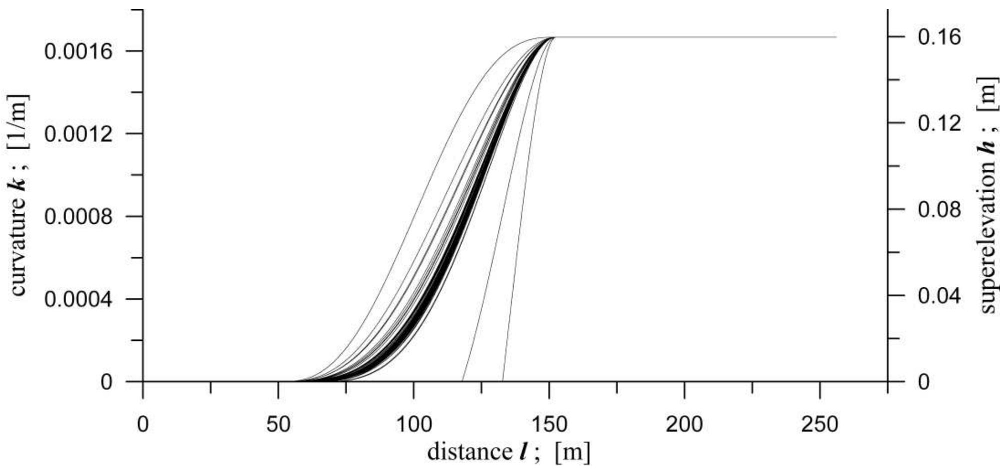

Fig. 4. Search for optimum 7th order TC's shape represented by curvature k and superelevation ramp. Condition for the tangence of the k and h is imposed. QF_2 is used

5.2. Results of optimisation and dynamical simulations

In order to present the results of the optimisation and the corresponding results of simulations the 9th order TCs were chosen. The results are shown in Figs. 5-9.

It can be seen in Figs. 5 and 6 that for different QFs optimisation process goes differently. Different sets of the curves are tested for being or not the optimum ones. Obviously different sets of A'_i coefficients for optimum shapes are obtained (see Sec. 6), too. Figure 7 represents optimised shape of the curves and features for such curves. Figure 7(a) represents the shapes in plane while Fig. 7(b) the shape of superelevation ramps. Basic features of the curves are shown in Figs. 7(b) and 7(c). Figure 7(b) shows curvatures and Fig. 7(c) shows superelevation ramp slopes, respectively. Denotations used in these figures are OPF2, OPF7, and INIT for QF_2, QF_7 quality functions, and for the initial shape given with Eq. (28), respectively. Figures 8 and 9 represent selected dynamical quantities that show improvement in the vehicle dynamical behaviour due to use of the optimised TC shape. Comparison for the initial (INIT) and the optimised (OPF7) curves is done there. Despite relatively small differences in these curves shapes and in their features seen in Fig. 7, the improvements revealed in Figs. 8 and 9, especially for the car body, are spectacular. Note, that car body lateral displacements y_b and accelerations $\ddot{y}_b$ for the optimised curve (OPF7) are much smaller than for the initial one (INIT). Differences for the wheelsets also exist but are much smaller, thus of much lower importance.

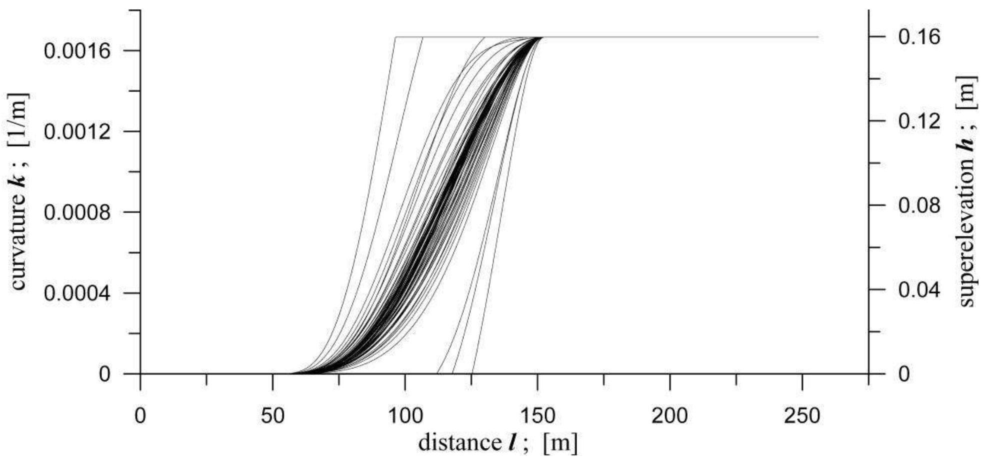

Fig. 5. Search for optimum 9th order TC's shape represented by curvature k and superelevation ramp. Condition for the tangence of the k and h is imposed. QF_2 is used

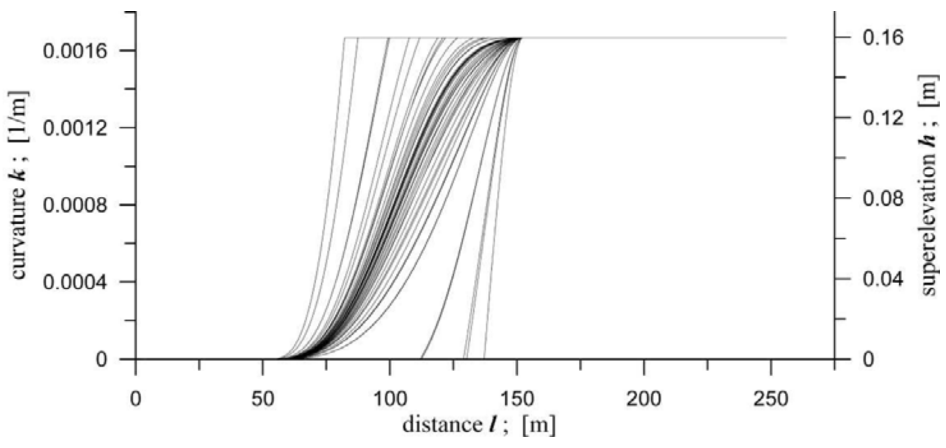

Fig. 6. Search for optimum 9th order TC's shape represented by curvature k and superelevation ramp. Conditions for the tangence of the k and h is imposed. QF_7 is used

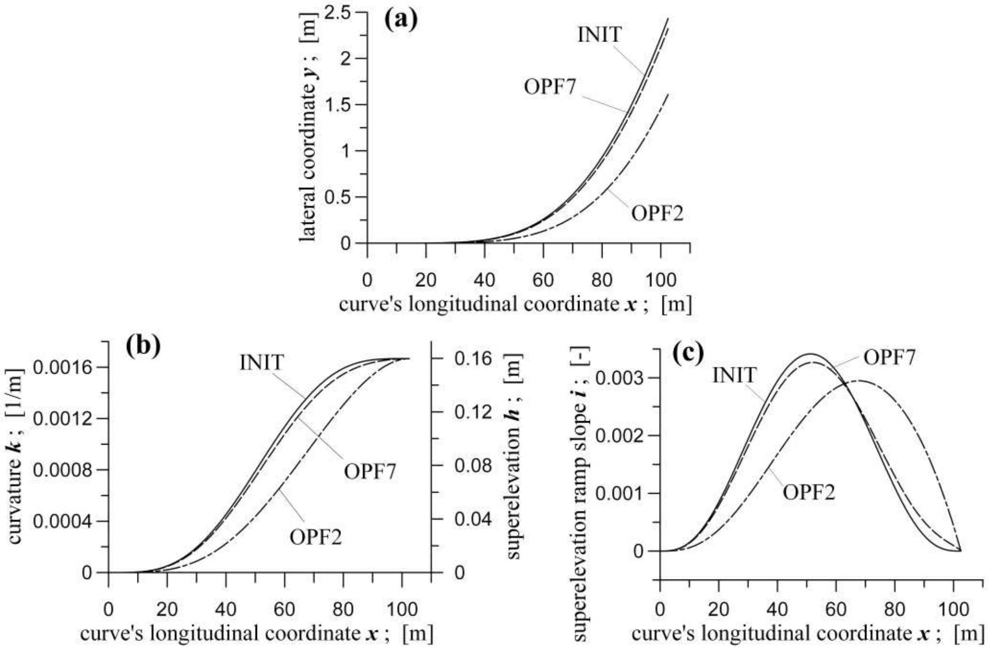

Fig. 7. Results of optimisation for QF_2 and QF_7 compared to the initial curve: (a) curve shapes, (b) curvatures and supelevations, (c) superelevation ramp slopes

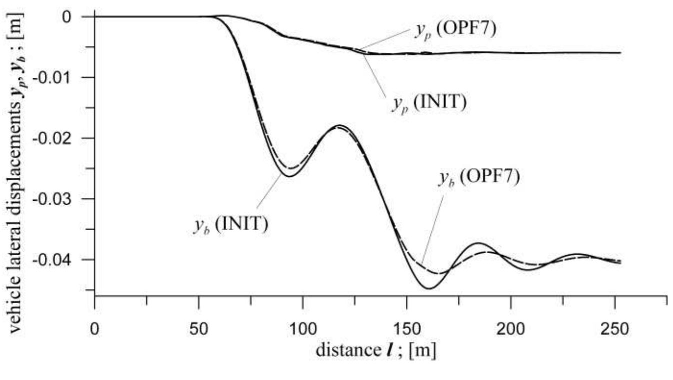

Fig. 8. Vehicle model lateral displacements for optimised (QF_7) and initial TCs

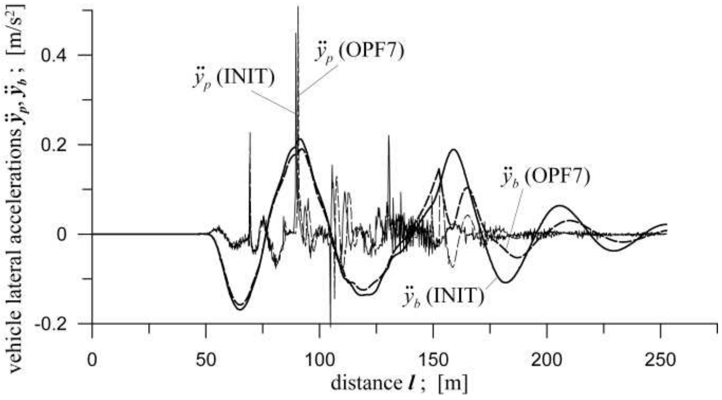

Fig. 9. Vehicle model lateral accelerations for optimised (QF_7) and initial TCs

6. Conclusion

The idea and the corresponding software presented in current paper appeared to be effective tool in optimising polynomial TCs form. It proved true for different quality functions based on the dynamical quantities taken from the advanced model of vehicle-track system. It was possible to generate A_i' polynomial coefficients in all case studies done so far. The sets of coefficients corresponding to TCs optimised in present paper, recorded from the highest to the lowest order of the terms, are as follows: Fig. 3, {0.19999, -0.49975, 0.37363}; Fig. 4, {1.3015, 2.4063, -0.26081, 0.0081477}; Fig. 5, {-0.26498, 1,3646, -2.736, 2.0801}; Fig. 6, {-0.28031, 1.25, -1.9939, 1.1667}. Number of the terms depends on the data taken before each optimisation and the demands adopted for the case considered. To get the TCs' equations, the above coefficients have to be introduced into Eq. (7).

It was shown that TCs properties can be improved by omission the lowest order terms. Also the demands imposed on the TCs'properties that have to be implemented in the code are important for such improvements. The example are the demands specified in Subsec. 3.1.

In the foreseeable future the following issues deserve our attention. More QFs should be tested and proposed. Some could take account of the several factors with use of the weighting factors, some could be calculated for the selected parts of TCs. Next, optimisations for the exit TCs and for the routes that include both the entrance and exit TCs would be desirable. Our software is capable of doing these. Number of the case studies necessary to perform in order to recommend any optimum TC shape for practice is inevitably large.

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